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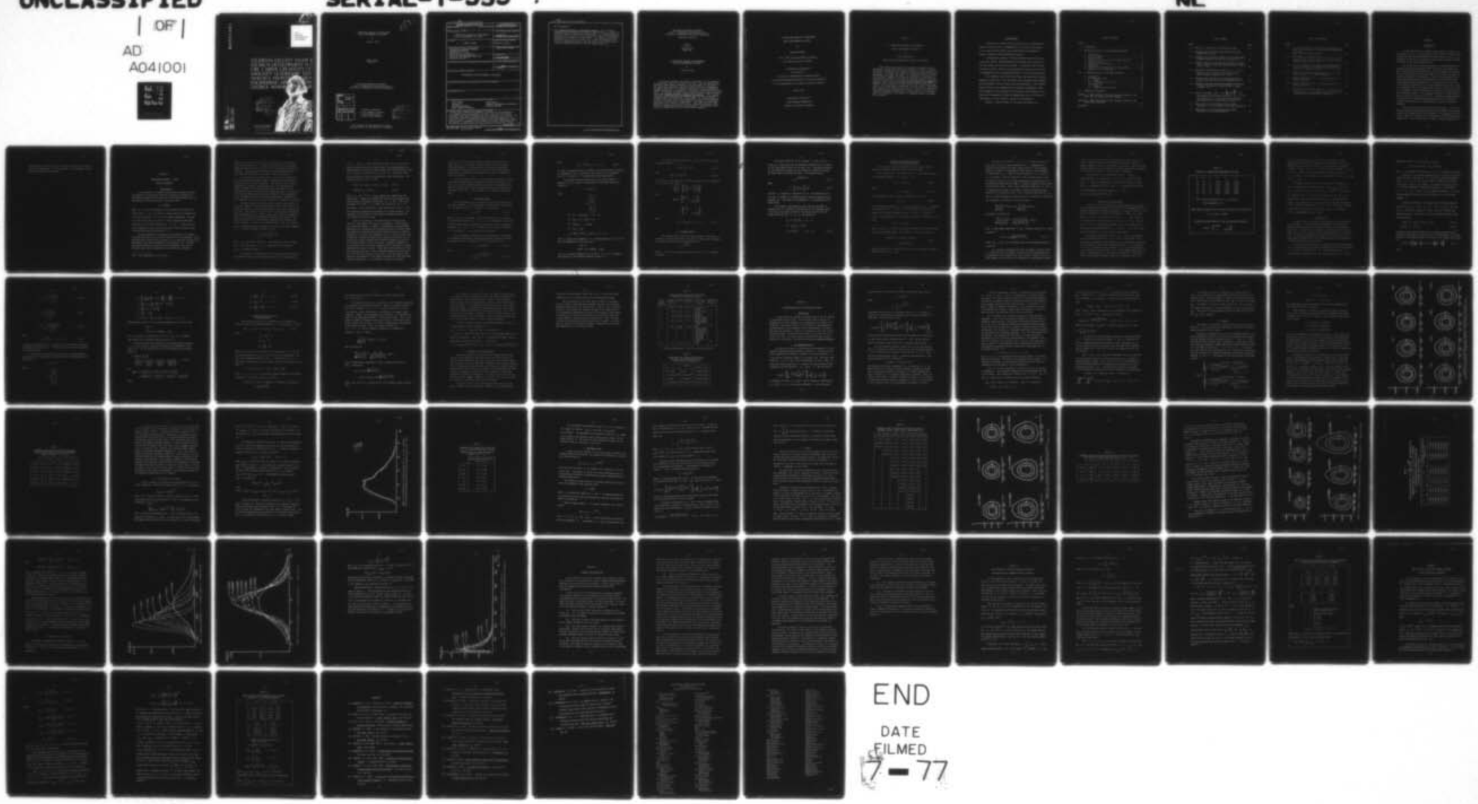

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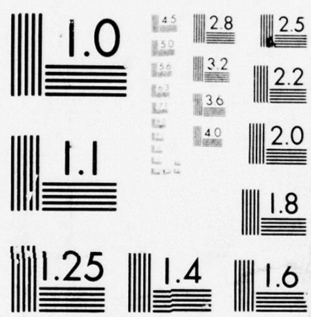
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A STATISTICAL ANALYSIS OF SOME MODELS
USED IN ACCELERATED LIFE TESTS

by

Henry D. Kahn

Serial T-353
8 May 1977

The George Washington University
School of Engineering and Applied Science
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This paper considers the problem of inference for the parameters of three models used in accelerated life tests. The first part obtains orthogonal least squares estimators for the parameters of the models. The second part considers a Bayesian analysis of the two parameters of the power rule model. The absolutely continuous bivariate exponential (ACBVE) distribution is assigned as a joint prior on the power rule parameters. The analysis proceeds along the lines dictated by Box and Tiao in their next page

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A Dissertation submitted to

the Faculty of

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May 8, 1977

Dissertation directed by

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Professor of Operations Research

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Nozer Darabsha Singpurwalla, Director of Research

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CHAPTER I

INTRODUCTION

In this dissertation we consider some problems of inference for accelerated life tests. Accelerated life tests are used to obtain failure data on items of high reliability in short periods of time. An introduction to the topic of accelerated life testing is given in Chapter 9 of Mann, Schafer and Singpurwalla [11].

For the purposes of this dissertation we consider parametric accelerated life testing procedures which assume that items under test have a particular failure distribution. We also assume that the parameters of this failure distribution change with the stress according to a prescribed functional relationship. That is, the parameters of the failure distribution may be re-parameterized in terms of the stress. The re-parameterization appropriate for a particular life test is derived through consideration of the failure mechanism of the test items. It is understood that a re-parameterization model is valid over a range of environmental stress that includes use conditions, or normal stress, and accelerated stresses. The accelerated stresses are large enough to induce failure in a short period of time. A similar situation prevails in certain kinds of biomedical experiments wherein the parameters of survival distributions are re-parameterized in terms of concomitant variables.

This dissertation is divided into two main parts. In the first part we consider a non-Bayesian approach for the analysis of three models which re-parameterize the scale parameter of an exponential distribution. These are the power rule, the Arrhenius, and the Eyring models, respectively. In the second part we consider a Bayesian approach for

the analysis of accelerated life test data in which the power rule is used to re-parameterize the scale parameter of an exponential distribution and a Weibull distribution.

CHAPTER II

NON-BAYESIAN ANALYSIS: LEAST SQUARES ESTIMATION

1. Introduction

In this chapter, we consider least squares estimators for some accelerated life test models involving an exponential distribution. The models considered are the power rule, the Arrhenius model, and the Eyring model, and may be written in the general form

$$\Theta_i = \exp(\underset{\sim}{X}_i \underset{\sim}{\beta})$$

where Θ_i is the scale parameter of the exponential distribution at stress level i, $\underset{\sim}{X}_i$ is a vector of stress values that represent the ith stress level and $\underset{\sim}{\beta}$ is a vector of unknown parameters. The general form of this function leads to a linear regression model which expresses the relationship between $\log \Theta_i$† and $\underset{\sim}{\beta}$. Data from life tests conducted at several accelerated stress levels may then be used to obtain least squares estimators of $\underset{\sim}{\beta}$.

Various approaches for estimating the parameters of the models considered here have been studied for different life distributions. Maximum likelihood estimators for the power rule are presented by Nelson [12], Singpurwalla [19] and by Singpurwalla and Al-Khayyal [21]. The maximum likelihood approach was also used by Singpurwalla [20] for the Arrhenius model with an exponential life distribution. Graphical

†Note: All logarithms are to the base e.

methods for the Arrhenius model with a log normal life distribution were given by Nelson [13]. In another paper, Nelson [14] presented graphical methods for the power rule with a two parameter Weibull distribution. Linear regression methods for the Arrhenius model with a log normal life distribution were given by Nelson and Hahn [16].

In the approach we consider here, we define linear models for the re-parameterization models using a log gamma error distribution and allow the use of censored samples. This approach follows that of Singpurwalla, Castellino and Goldschen [22]. They obtained least squares estimates of the Eyring parameters and assumed them to be approximately normal on the basis of a simulation study. In this chapter we obtain orthogonal least squares estimators for the parameters of all the models considered and establish approximate normality for the estimators in terms of their exact measures of skewness and kurtosis.

The methods presented in this chapter depend on the results of life tests conducted in the following manner. The ith life test is conducted by placing n_i items on test with the stress level fixed at the ith stress. The sequence of stress levels at which tests are conducted and the assignment of test items to stress levels should be randomized. This is done to help insure independence between test results at the various stresses and to avoid possible systematic bias entering into the test results. If the test is terminated after $r_i (\leq n_i)$ failures occurs, a minimum variance, unbiased, efficient and sufficient estimator of Θ_i is given by Epstein and Sobel [6] as

$$\hat{\Theta}_i = \sum_{j=1}^{r_i} \frac{t_{ij} + (n_i - r_i)\, t_{ir_i}}{r_i}$$

where t_{ij} = jth failure time $(j=1,\ldots,r_i)$ observed at the ith stress level. The distribution of $\hat{\Theta}_i$ is a gamma with scale parameter (r_i/Θ_i) and shape parameter r_i.

Linear models of observations for the re-parameterization models considered here may then be formulated in terms of the random variable

log $\hat{\Theta}_i$. Since $\hat{\Theta}_i$ has a gamma distribution, log $\hat{\Theta}_i$ has a log gamma distribution which for large values of the shape parameter is approximated by a normal distribution (see Johnson and Kotz [8] p. 196). Thus, a linear model based on log $\hat{\Theta}_i$ with r_i sufficiently large may be considered to have an approximately normal distribution. The mean and variance of log $\hat{\Theta}_i$ are

$$E(\log \hat{\Theta}_i) = \log \Theta_i - \log r_i + \psi(r_i) \qquad (2.1.1)$$

$$Var(\log \hat{\Theta}_i) = \psi^{(')}(r_i) \qquad (2.1.2)$$

where $\psi(\cdot)$ and $\psi^{(')}(\cdot)$ are the digamma and trigamma functions, respectively. Digamma and trigamma functions may be evaluated using formulas given by Abromowitz and Stegum [1], p. 258). Thus, when the number of failures r_i is known, the variance of log $\hat{\Theta}_i$ is also known. We may then define a linear model based on log $\hat{\Theta}_i$ with the observations appropriately weighted by the known variances. The variances and covariances of least squares estimates obtained from such a model will then be known.

Since the variances of the observations are known, we may construct linear models that result in orthogonal, or uncorrelated, least squares estimates. A number of investigators, using different approaches, have determined that orthogonal designs for linear regression models have desirable and optimal properties (see, for example, Ehrenfeld [5], Elfving [6], Himmelblau [7], pp. 234-237, Tocher [23]. In addition, orthogonal designs greatly simplify the process of computing least squares estimates and circumvent problems that arise if the covariance matrix is ill-conditioned. Orthogonal estimates for the accelerated life test model parameters may be obtained by slightly modifying the form of the model. These modifications make use of the known variance structure of the corresponding regression model and do not alter the basic character of the model. The least squares estimates for the parameters of the models considered here have some other convenient properties: it is possible to obtain least squares

estimates that are best linear unbiased and uncorrelated for any sample size, i.e., any number of stress levels; the exact variance and higher moments of the estimates may be obtained; approximate normality may be established for the estimates for a small number of stress levels, provided a moderate number of failures is observed at each stress level.

In the following sections of this chapter we consider the power rule, Arrhenius and Eyring re-parameterization models in detail. In each case we define regression models that result in orthogonal least squares estimates for the model parameters and develop procedures for inference at use stress. The procedures are illustrated using some data.

2. Power Rule Model

For an exponential time to failure the power rule assumes that the scale parameter Θ is proportional to the Pth power of the stress V. The power rule for Θ at stress i is written as

$$\Theta_i = C/V_i^P \qquad (2.2.1)$$

where C and P are unknowns. Life tests conducted at k distinct accelerated stress levels V_i, $i = 1,\ldots,k$, as described in the first section of this chapter, provide data that may be used to compute the estimates $\hat{\Theta}_i$, $i = 1,\ldots,k$. This data may also be used to estimate C, P and predict Θ_u, the mean life at use stress.

We may formulate a linear model for the power rule that results in orthogonal least squares estimates by slightly modifying the model of Equation (2.2.1). The modified form of the power rule is

$$\Theta_i = C^*/(V_i/\bar{V})^P \qquad (2.2.2)$$

where

$$\bar{V} = \prod_{i=1}^{k} V_i^{\,w_i^{-1}\Big/\sum_{i=1}^{k} w_i^{-1}} \qquad (2.2.3)$$

with

$$w_i = \psi^{(\prime)}(r_i) \ , \ i = 1,\ldots,k \ . \qquad (2.2.4)$$

In effect, the modification is a change in the proportionality constant C in Equation (2.2.1), indicated by the notation C^* in Equation (2.2.2). The value of P is unaffected by the modification.

If we assume a multiplicative error term for the model of Equation (2.2.2), a model of observations for the power rule may be written as

$$\underset{\sim}{Z} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{e}$$

where

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \\ \vdots \\ \underset{\sim}{X}_k \end{bmatrix}$$

$$\underset{\sim}{X}_i = [1 \ , \ - \log V_i/\overline{V}] \ , \ i = 1,\ldots,k$$

$$\underset{\sim}{\beta}' = [\beta_0 \ , \ \beta_1] \approx [\log C^* \ , \ P]$$

$$\underset{\sim}{e}' = [\log e_1 \ ,\ldots, \ \log e_k]$$

$$\underset{\sim}{Z}' = [z_1,\ldots,z_k]$$

$$z_i = \log \hat{\Theta}_i + \log r_i - \psi(r_i) \ , \ i = 1,\ldots,k$$

and $\underset{\sim}{\beta}'$ denotes the transpose of $\underset{\sim}{\beta}$. From Equations (2.1.1), (2.1.2) and the randomization of the life test

$$E(\underset{\sim}{e}) = 0$$

$$E(\underset{\sim}{e}'\underset{\sim}{e}) = \underset{\sim}{W} = \text{diag}[w_1,\ldots,w_k]$$

The w_i are given by Equation (2.2.4) and $\underset{\sim}{W}$ is a $k \times k$ diagonal matrix which is the variance-covariance matrix of $\underset{\sim}{e}$.

By the Gauss-Markov Theorem, the least squares or best linear unbiased estimate of $\underset{\sim}{\beta}$ is

$$\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\, \underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{Z} \qquad (2.2.5)$$

and

$$\underset{\sim}{\mathrm{Var}}(\hat{\underset{\sim}{\beta}}) = (\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1} \quad . \qquad (2.2.6)$$

For the form of the power rule defined by Equation (2.2.2), Equations (2.2.5) and (2.2.6) reduce to

$$\hat{\underset{\sim}{\beta}} = \begin{pmatrix} \hat{\beta}_o \\ \hat{\beta}_1 \end{pmatrix} = \left[\frac{\Sigma w_i^{-1} z_i}{\Sigma w_i^{-1}} , \frac{\Sigma w_i^{-1} x_i z_i}{\Sigma w_i^{-1} x_i^2} \right]'$$

and

$$\underset{\sim}{\mathrm{Var}}(\hat{\underset{\sim}{\beta}}) = \begin{bmatrix} \mathrm{Var}(\hat{\beta}_o) & 0 \\ 0 & \mathrm{Var}(\hat{\beta}_1) \end{bmatrix}$$

$$= \begin{bmatrix} 1/\Sigma w_i^{-1} & 0 \\ 0 & 1/\Sigma w_i^{-1} x_i^2 \end{bmatrix}$$

where

$$x_i = -\log V_i/\bar{V} \ , \ i = 1,\ldots,k \ . \qquad (2.2.7)$$

3. Arrhenius Model

The Arrhenius model usually has applications in tests where temperature is the accelerating stress and it assumes that the exponential scale parameter may be re-parameterized as

$$\Theta_i = e^{A+B/T_i}$$

where i refers to a particular stress level and A, B are unknown constants.

The natural logarithm of the estimates $\hat{\Theta}_i$ may be used to construct a linear model for the Arrhenius parameters that is similar to the linear model defined for the power rule. Orthogonal estimates for the Arrhenius parameters may be obtained by slightly modifying the basic form of the Arrhenius model. The modified form is

$$\Theta_i = e^{A^* + B(T_i^{-1} - \bar{T})} , \qquad (2.3.1)$$

where

$$\bar{T} = \sum_{i=1}^{k} w_i^{-1} T_i \Big/ \sum_{i=1}^{k} w_i^{-1} , \qquad (2.3.2)$$

with the w_i defined as in Equation (2.2.3). The modification is, in effect, a change in the proportionality constant indicated by the notation A^* . The value of the parameter B is unaffected by the modification.

If we assume a *multiplicative error term for the model of* Equation (2.3.1), the model of observations for the Arrhenius model is the same as the model defined in the previous section for the power rule except that $\underset{\sim}{X}_i$, $\underset{\sim}{\beta}$, x_i are defined as

$$\underset{\sim}{X}_i = [1, (T_i^{-1} - \bar{T})] , \quad i=1,\dots,k$$

$$\underset{\sim}{\beta}' = [\beta_0, \beta_1] = [A^*, B]$$

$$x_i = (T_i^{-1} - \bar{T}) , \qquad i=1,\dots,k . \qquad (2.3.3)$$

4. Inference at Use Stress for the Power Rule and Arrhenius Models

The inference procedures presented in this section apply, in essence, to both the power rule and the Arrhenius model.

The log mean life at use stress, $\log \theta_u$, is a linear function of the unknown parameters $\underset{\sim}{\beta}' = (\beta_0, \beta_1)$, i.e.,

$$\log \theta_u = \beta_0 + \beta_1 x_u \tag{2.4.1}$$

where

$$x_u = - \log V_u / \bar{V} \tag{2.4.2}$$

for the power rule with $\bar{V}$ given by Equation (2.2.3) and

$$x_u = T_u^{-1} - \bar{T} \tag{2.4.3}$$

for the Arrhenius model with $\bar{T}$ given by Equation (2.3.2). In Equation (2.4.2) and (2.4.3), V_u and T_u are use stress, respectively. We obtain the best linear unbiased estimate of $\log \theta_u$, denoted by $\log \bar{\theta}_u$, by substituting the least squares estimates in Equation (2.4.1). Thus,

$$\log \bar{\theta}_u = \hat{\beta}_0 + \hat{\beta}_1 x_u ,$$

where $\hat{\beta}_0$ and $\hat{\beta}_1$ are the least squares estimates for either the power rule or the Arrhenius model. The variance of $\log \bar{\theta}_u$ is given by

$$\begin{aligned} \mathrm{Var}(\log \bar{\theta}_u) &= \mathrm{Var}(\hat{\beta}_0) + x_u \mathrm{Var}(\hat{\beta}_1) \\ &= \underset{\sim}{X}_u (\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X})^{-1} \underset{\sim}{X}_u' , \end{aligned} \tag{2.4.4}$$

where the bottom line of Equation (2.4.4) gives the variance in matrix notation with $\underset{\sim}{X}_u = [1 , x_u]$.

The exact distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$ would be difficult to obtain since they are linear combinations of k independent non-identically distributed log gamma random variables. The log gamma, however, is known to be approxmately normal when its shape parameter is sufficiently large. Since the shape parameter of z_i is equal to r_i, the number of failures observed at the ith stress level, we may consider the estimates to be linear combinations of approximately normal random variables when the number of failures observed at each stress is sufficiently large. Thus, we may reasonably assume that $\hat{\beta}_0$ and $\hat{\beta}_1$ are approximately normal if the number of failures observed at each stress level is moderately large. Also, it has been verified that $\hat{\beta}_0$ and $\hat{\beta}_1$ satisfy Lyapunov's condition (see, e.g., Parzen [17], page 432) for convergence to normality so that for large k

$$\frac{\hat{\beta}_0 - \beta_0}{\sqrt{\mathrm{Var}(\hat{\beta}_0)}} \mathrel{\dot{\sim}} N(0,1) \quad \text{and} \quad \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\mathrm{Var}(\hat{\beta}_1)}} \mathrel{\dot{\sim}} N(0,1) \ ,$$

from which it follows that

$$\frac{\log \bar{\Theta}_u - \log \Theta_u}{\sqrt{\mathrm{Var}(\log \bar{\Theta}_u)}} = \frac{\log \bar{\Theta}_u - \log \Theta_u}{\sqrt{\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'}} \mathrel{\dot{\sim}} N(0,1) \ .$$

Thus, a large sample approximate $(1-\alpha)\%$ confidence interval for $\log \bar{\Theta}_u$ is given by

$$\log \bar{\Theta}_u \pm Z_{\alpha/2}\sqrt{\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'} \ , \qquad (2.4.5)$$

where $Z_{\alpha/2}$ is the $\alpha/2$ percentage point of the standard normal distribution.

In a typical accelerated life test, the number of stress levels k is small and hence the degree to which the normal approximates the distribution of the estimators may be of concern. However, even for a small number of stress levels, the estimators are approximately normal

in the sense that their skewness and kurtosis, or shape factors, are close to zero, the values for the normal distribution, provided a moderate number of failures is observed at each stress. Also, the shape factors approach zero as the number of failures observed and the number of stress levels increases. This is shown in Appendix A.

We note that if it is assumed that $\log \bar{\Theta}_u \overset{\bullet}{\sim} N(\log \Theta_u ,$ $\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u')$, it follows that $\bar{\Theta}_u \overset{\bullet}{\sim} \Lambda(\log \Theta_u , \underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u')$ where $\overset{\bullet}{\sim} \Lambda$ denotes "approximately log normally distributed." Thus $E(\bar{\Theta}_u) \overset{\sim}{\sim} \Theta_u \exp[\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'/2]$, and $\tilde{\Theta}_u = \bar{\Theta}_u \exp[-\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'/2]$ is an approximately unbiased estimate of Θ_u with $\mathrm{Var}(\tilde{\Theta}_u) = \Theta_u^2\{\exp[\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u']-1\}$.

5. Example for the Power Rule

In this section, we illustrate procedures presented in previous sections using data simulated for the power rule by Singpurwalla [19]. An example involving the Arrhenius model would, of course, be similar. The data was generated on a computer by choosing $C = 1000$, $P = 3$ and using $\Theta_i = CV_i^{-P}$ to calculate five values for Θ_i . The results of the simulation are shown in Table 2.1 with the n_i and r_i used for each V_i and $\hat{\Theta}_i$ determined by Equation (2.1). Also shown are the approximately unbiased estimate of Θ_i , and the $\underset{\sim}{\hat{\beta}}$ and $\underset{\sim}{V}(\underset{\sim}{\hat{\beta}})$ determined by Equations (2.2.5) and (2.2.6), respectively.

The maximum likelihood estimate of P , 3.09, is comparable to the least squares estimate of P , $\hat{\beta}_1 = 3.05$. The variance of the least squares estimate is 2.94% greater than the Cramér-Rao lower bound on the variance of the maximum likelihood estimate. The estimates of C^* are not comparable because the value of C^* in the least squares case is different from the value of C^* in the maximum likelihood case. That is, the modification of the power rule required to obtain uncorrelated least squares estimates results in a different value of C^*

TABLE 2.1

RESULTS OF A MONTE CARLO ACCELERATED LIFE TEST

V_i	n_i	r_i	Θ_i	$\hat{\Theta}_i$	$\tilde{\Theta}_i$
10	30	15	1.000	1.308	0.998
20	30	15	0.125	0.078	0.123
30	30	20	0.037	0.030	0.036
40	30	25	0.016	0.017	0.015
50	30	25	0.008	0.008	0.008

Note: With the exception of $\tilde{\Theta}_i$, this data is from Singpurwalla [19].

Least Squares Estimates for the Power Rule Based on the Above Data:

$$\hat{\underset{\sim}{\beta}}' = (-3.2526\ ,\ 3.0464)\ .$$

Variance-covariance Matrix of the Least Squares Estimates:

$$\mathrm{Var}(\hat{\underset{\sim}{\beta}}) = \begin{bmatrix} 0.01025 & 0 \\ 0 & 0.03538 \end{bmatrix}.$$

from that which results when the power rule is modified to yield asymptotically uncorrelated maximum likelihood estimates. Also, the maximum likelihood procedure estimates C^* while the least squares estimates $\log C^*$. For $V_u = 7$, the least squares estimate of $\log \Theta_u$ is $\log \bar{\Theta}_u = 1.1105$ and the approximately unbiased estimate of Θ_u is $\tilde{\Theta}_u = 2.9126$. These compare to the actual values of $\log \Theta_u = 1.0700$ and $\Theta_u = 2.9155$.

The skewness and kurtosis of the least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ obtained from the data in Table 2.1 are $\gamma_{1,\hat{\beta}_0} = -0.1012$, $\gamma_{2,\hat{\beta}_0} = 0.0205$ and $\gamma_{1,\hat{\beta}_1} = -0.1552$, $\gamma_{2,\hat{\beta}_1} = 0.0549$. These values are reasonably close to zero so that we may assume $\hat{\beta}_0$ and $\hat{\beta}_1$ are approximately normal and it follows that $\log \bar{\Theta}_u$ is approximately normal. Thus, Expression (2.4.5) may be used to construct approximate $(1-\alpha)\%$ confidence intervals for $\log \Theta_u$. With $\mathrm{Var}(\log \bar{\Theta}_u) = 0.0828$, an approximate 95% confidence interval for $\log \Theta_u$ is obtained as $1.1105 \pm 1.96\sqrt{0.0828} = [0.5465, 1.6745]$.

6. Eyring Model

In this section we consider the Eyring model for an exponential scale parameter. In a recent paper, Singpurwalla, Castellino and Goldschen [22], obtained least squares estimates for the parameters of this model. In this section, we present a modification of the Eyring model that results in orthogonal least squares estimators for its parameters. Since the regression model defined in [22] results in an ill-conditioned covariance matrix, an orthogonal design for the estimation of the Eyring parameters is desirable.

An exponential distribution was assumed in [22] and a generalized form of the Eyring model was used to re-parameterize the scale parameter λ_i , in terms of a thermal stress T_i and non-thermal stress V_i . The Eyring model assumes that the time rate degradation of λ_i may be

expressed in terms of the stress pair (T_i, V_i) as

$$\lambda_i = AT_i \exp [-B/KT_i + CV_i + DV_i/KT_i]$$

where K is Boltzmann's constant and A, B, C, D are unknown parameters.

Estimates of the unknown parameters may be obtained from censored sample life tests conducted at $k > 4$ different stress environments $\{T_i, V_i\}$, $i = 1,\ldots,k$. The ith life test is conducted as described in the first section of this chapter. An estimate of the hazard rate λ_i may then be obtained by taking the reciprocal of the estimate $\hat{\Theta}_i$, i.e. $\hat{\lambda}_i = \hat{\Theta}_i^{-1}$. Since $\hat{\Theta}_i$ has a gamma distribution, $\hat{\lambda}_i$ has an inverted gamma distribution with scale parameter (r_i/Θ_i) and shape parameter r_i.

We may now formulate a linear model for the Eyring parameters in terms of $\log \hat{\lambda}_i$. Since $\hat{\lambda}_i = \hat{\Theta}_i^{-1}$, $\log \hat{\lambda}_i = -\log \hat{\Theta}_i$ and since $\log \hat{\lambda}_i$ is merely the negative of $\log \hat{\Theta}_i$, the two variables have similar properties. Thus, $\log \hat{\lambda}_i$ has a log gamma distribution with mean and variance

$$E(\log \hat{\lambda}_i) = \log \lambda_i + \log r_i - \psi(r_i) \qquad (2.6.1)$$

$$\mathrm{Var}(\log \hat{\lambda}_i) = \psi^{(')}(r_i) \,. \qquad (2.6.2)$$

Orthogonal least squares estimates for the Eyring parameters may be obtained by modifying the original form of the Eyring model. The basic character of the modified model is the same as the original and is given by

$$\lambda_i = A^*T_i \exp\left[-B\left(\frac{1}{KT_i} - \overline{T}\right) + C(V_i - \overline{V}) + D\left(\frac{V_i}{KT_i} - \overline{VT}\right)\right] \qquad (2.6.3)$$

where

$$\bar{T} = \sum_{j=1}^{k} \frac{w_j^{-1} (KT_j)^{-1}}{\sum_{j=1}^{k} w_j^{-1}} \qquad (2.6.4)$$

$$\bar{V} = \sum_{j=1}^{k} \frac{w_j^{-1} V_j}{\sum_{j=1}^{k} w_j^{-1}} \qquad (2.6.5)$$

$$\overline{VT} = \sum_{j=1}^{k} \frac{w_j^{-1} (V_j/KT_j)^{-1}}{\sum_{j=1}^{k} w_j^{-1}} \qquad (2.6.6)$$

with

$$w_j = \psi^{(')}(r_j) \text{ , } j = 1,\ldots,k \text{ .} \qquad (2.6.7)$$

In effect, the modification is a change in the proportionality constant indicated by the notation A^* . The values of B, C and D are unaffected by the modification.

If we assume a multiplicative error term for the model given by Equation (2.6.3) a linear model of observation for the Eyring model can be written as

$$\underset{\sim}{Z} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{e}$$

where

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \\ \vdots \\ \underset{\sim}{X}_k \end{bmatrix}$$

$$X_i = \left[1 , \left(\frac{1}{KT_i} - \overline{T}\right) , (V_i - \overline{V}) , \left(\frac{V_i}{KT_i} - \overline{VT}\right)\right] , \quad i = 1,\ldots,k$$

$$\beta' = \left[\beta_1, \beta_2, \beta_3, \beta_4\right] = \left[\log A^* , -B, C, D\right]$$

$$e' = \left[\log e_1, \ldots, \log e_k\right]$$

$$Z' = \left[z_1, \ldots, z_k\right]$$

$$z_i = \log \hat{\lambda}_i - \log r_i + \psi(r_i) - \log T_i , \quad i = 1,\ldots,k .$$

From Equations (2.6.1), (2.6.2) and the randomization of the life test

$$E(e) = 0$$

$$E(e'e) = W = \mathrm{diag}[w_1, \ldots, w_k] .$$

The w_j are given by Equation (2.6.7) and W is a $k \times k$ diagonal matrix which is the variance-covariance matrix of e .

Equations for the least squares or best linear unbiased estimate of β , $\hat{\beta}$ and $\mathrm{Var}(\hat{\beta})$, are given in matrix form by Equations (2.2.4) and (2.2.5). For the modified form of the Eyring model these equations reduce to

$$\hat{\beta} = \left(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4\right)'$$

$$= \left[\frac{\Sigma w_i^{-1} z_i}{\Sigma w_i^{-1}} , \frac{\Sigma x_{i2} w_i^{-1} z_i}{\Sigma x_{i2}^2 w_i^{-1}} , \frac{\Sigma x_{i3} w_i^{-1} z_i}{\Sigma x_{i3}^2 w_i^{-1}} , \frac{\Sigma x_{i4} w_i^{-1} z_i}{\Sigma x_{i4}^2 w_i^{-1}}\right] \tag{2.6.8}$$

and

$$\mathrm{Var}(\hat{\beta}) = \mathrm{diag}\left[\mathrm{Var}(\hat{\beta}_1), \mathrm{Var}(\hat{\beta}_2), \mathrm{Var}(\hat{\beta}_3), \mathrm{Var}(\hat{\beta}_4)\right]$$

$$= \mathrm{diag}\left[(\Sigma w_i^{-1})^{-1}, (\Sigma x_{i2}^2 w_i^{-1})^{-1}, (\Sigma x_{i3}^2 w_i^{-1})^{-1}, (\Sigma x_{i4}^2 w_i^{-1})\right]$$

where

$$x_{i2} = \left(\frac{1}{KT_i} - \overline{T}\right), \quad i = 1,\ldots,k \qquad (2.6.9)$$

$$x_{i3} = \left(V_i - \overline{V}\right), \quad i = 1,\ldots,k \qquad (2.6.10)$$

$$x_{i4} = \left(\frac{V_i}{KT_i} - \overline{VT}\right), \quad i = 1,\ldots,k \; . \qquad (2.6.11)$$

7. Inference at Use Stress for the Eyring Model

The log hazard rate at use conditions may be expressed as a linear function of the unknown parameters $\underset{\sim}{\beta}' = [\beta_1, \beta_2, \beta_3, \beta_4]$. That is

$$\log \lambda_u = \log T_u + \beta_1 + \beta_2 x_{2u} + \beta_3 x_{3u} + \beta_4 x_{4u} \qquad (2.7.1)$$

where

$$x_{2u} = 1/KT_u - \overline{T}$$

$$x_{3u} = V_u - \overline{V}$$

$$x_{4u} = \frac{V_u}{KT_u} - \overline{VT}$$

where (T_u, V_u) are the use condition stress values and $\overline{T}$, $\overline{V}$ and $\overline{VT}$ are given by Equations (2.6.4), (2.6.5) and (2.6.6) respectively. The best linear unbiased estimate of log λ_u , denoted by log $\overline{\lambda}_u$, is obtained by substituting the least squares estimates in Equation (2.7.1). That is,

$$\log \overline{\lambda}_u = \log T_u + \hat{\beta}_1 + \hat{\beta}_2 x_{2u} + \hat{\beta}_3 x_{3u} + \hat{\beta}_4 x_{4u}$$

where $\hat{\beta}_i$, $i = 1,\ldots,4$, are the least squares estimates of the Eyring parameters. The variance of log $\overline{\lambda}_u$ is then

$$\begin{aligned} \mathrm{Var}(\mathrm{iog}\ \overline{\lambda}_u) &= \mathrm{Var}(\hat{\beta}_1) + x_{2u}^2 \mathrm{Var}(\hat{\beta}_2) + x_{3u}^2 \mathrm{Var}(\hat{\beta}_3) + x_{4u}^2 \mathrm{Var}(\hat{\beta}_4) \\ &= \underset{\sim}{X}_u (\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X})^{-1} \underset{\sim}{X}_u \end{aligned}$$

where the second line gives the variance in matrix notation with $\underset{\sim}{X}_u = [1 \; x_{2u} \; x_{3u} \; x_{4u}]$.

The situation in this case is similar to that considered previously for the power rule and Arrhenius models. That is, the exact distribution of the estimates $\hat{\beta}_i$, $i = 1,\ldots,4$ would be difficult to obtain since they are linear combinations of non-identical log gamma random variables. However, the log gamma may be considered to be approximately normal if the gamma shape parameter is sufficiently large. Thus, if the number of failures observed at each stress level is sufficiently large, we may consider the estimates to be linear combinations of approximately normal random variables. Also, it has been verified that $\hat{\beta}_i$, $i = 1,\ldots,4$ satisfy Lyapunov's condition for convergence to normality. Thus, for large k

$$\frac{\hat{\beta}_i - \beta_i}{\sqrt{\mathrm{Var}(\hat{\beta}_i)}} \overset{\cdot}{\sim} N(0,1) \, , \; i = 1,2,3,4$$

and it follows that

$$\frac{\log \bar{\lambda}_u - \log \lambda_u}{\sqrt{\mathrm{Var}(\log \bar{\lambda}_u)}} = \frac{\underset{\sim}{X}_u \hat{\underset{\sim}{\beta}} - \underset{\sim}{X}_u \underset{\sim}{\beta}}{\sqrt{\underset{\sim}{X}_u (\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X})^{-1} \underset{\sim}{X}_u'}} \overset{\cdot}{\sim} N(0,1) \quad .$$

Thus, a large sample, approximate $(1-\alpha)\%$ confidence interval for $\log \lambda_u$ is given by

$$\log \bar{\lambda}_u \pm Z\alpha/2 \sqrt{\mathrm{Var}(\log \bar{\lambda}_u)}$$

$$= \underset{\sim}{X}_u \underset{\sim}{\beta} + \log T_u \pm Z\alpha/2 \sqrt{\underset{\sim}{X}_u (\underset{\sim}{X}' \underset{\sim}{W}^{-1} \underset{\sim}{X})^{-1} \underset{\sim}{X}_u'}$$

where $Z\alpha/2$ is the $\alpha/2$ percent point of the standard normal distribution.

In a typical accelerated life test, the number of stress levels k is small and hence the degree to which the normal approximates the distribution of the estimates may be of concern. We may, however, obtain estimates that are approximately normal even for a small number of stress levels. If the number of failures observed at each stress is moderately large, the estimates are approximately normal in the sense that their skewness and kurtosis, or shape factors, are close to zero, the values for the normal distribution. Also, the shape factors of the estimates approach zero as the number of failures observed and the number of stress levels increase. This is shown in Appendix B.

We note that when the best linear unbiased estimate of $\log \bar{\lambda}_u$ is approximately normally distributed, i.e.,

$\log \bar{\lambda}_u \overset{\bullet}{\sim} N(\log \lambda_u\ ,\ \underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u')$, it follows that

$\bar{\lambda}_u \overset{\bullet}{\sim} \Lambda(\log \lambda_u\ ,\ \underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u')$ where "$\overset{\bullet}{\sim} \Lambda$" denotes "approximately log normally distributed." Thus, $E(\bar{\lambda}_u) \overset{\sim}{\sim} \lambda_u \exp[\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'/2]$ and an approximately unbiased estimate of λ_u is given by

$\tilde{\lambda}_u = \bar{\lambda}_u \exp[-\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u'/2]$ with $\mathrm{Var}(\tilde{\lambda}_u) = \lambda_u\{\exp[\underset{\sim}{X}_u(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}_u']-1\}$.

8. Example for the Eyring Model

Some accelerated life test data for test items which were subjected to both temperature and voltage stress is given in Table 2.2. This data was published in [22] and is used here to illustrate estimation procedures for the Eyring model. Orthogonal least squares estimates for the Eyring model parameters based on this data are shown in Table 2.3 along with the variances and shape factors. The equations for the shape factors, or skewness and kurtosis, of the estimates for the Eyring parameters are derived in Appendix B. The values of the shape factors for the estimates, shown in Table 2.3, are fairly close to zero so that an assumption of normality appears to be reasonable.

Use stress values for the items used to obtain the life test data of Table 2.2 were not available. However, if we assume that the

estimates are approximately normal then $\log \bar{\lambda}_u$ or any other linear combination of the estimates would also be approximately normal.

The accelerated life test data shown in Table 2.2 illustrate a point in regard to measuring the approximate normality of the Eyring parameter estimates by their shape factors. That is, since the shape factors of the estimates are weighted averages of the shape factors of the individual observations, the effect of obtaining a small number of failures at one or more stress levels may be offset by observing a larger number of failures at other stress levels. Thus, the shape factors of the estimates obtained from the data in Table 2.2 approximate the values for the normal distribution although only one failure was observed at two of the stress levels.

TABLE 2.2

TANTALUM CAPACITOR FAILURE TIMES UNDER
VOLTAGE AND TEMPERATURE STRESSES

Test Number (Index i)	Number on Test n_i	Voltage V_i	Temperature °C T_i	Actual Times to Failure (hr)	Number of Failures r_i
1	1,000	35.0V	85 °C	20,90,700, 37000	4
2	200	40.6V	85 °C	20,3600,9500, 27000	4
3	50	46.5V	85 °C	800,2800	2
4	50	51.5V	85 °C	500,800,2400, 10700, 10700	5
5	500	46.5V	45 °C	100,1200,7500, 20000,26000, 27300	6
6	175	46.5V	5 °C	1000	1
7	175	62.5V	5 °C	25,40,165,500, 620,720,820, 910,980,1270, 1600,2270, 2370,4590, 4690,4880. 7560,8730, 12500	19
8	50	57.0V	45 °C	8900	1

Data from Singpurwalla, Castellino and Goldschen [22].

TABLE 2.3

ORTHOGONAL LEAST SQUARES ESTIMATES FOR
THE EYRING MODEL PARAMETERS BASED
ON THE TEST DATA OF TABLE 2.2

i	β_i	$\mathrm{Var}(\hat{\beta}_i)$	$\gamma_{1,\hat{\beta}_i}$	$\gamma_{2,\hat{\beta}_i}$
1	-18.4933	0.0261	-0.1604	0.0512
2	0.4054×10^{-12}	0.3729×10^{-26}	-0.1789	0.0688
3	0.1543	0.2709×10^{-3}	-0.2032	0.0926
4	0.3870×10^{-14}	0.2062×10^{-30}	-0.1831	0.0715

CHAPTER III

A BAYESIAN ANALYSIS OF THE POWER RULE MODEL

1. Introduction

In this chapter we consider a Bayesian approach for the analysis of accelerated life test data. Here the power rule is used to re-parameterize the scale parameter of an exponential and a Weibull distribution. The approach we take is the following: Re-parameterize the distribution of the estimator of the scale parameter in terms of the power rule parameters. Specify an appropriate joint prior distribution and compute the posterior distribution using data from accelerated life tests. The joint posterior distribution then provides a solution to the problem of inference concerning the unknown parameters.

2. The Exponential Case

The power rule model for use with the scale parameter of an exponential distribution was described in Chapter II. Accelerated life tests, conducted as described in Chapter II, provide estimates of θ_i to be used for a Bayesian analysis of the power rule. The joint likelihood of $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)$ given the estimates $\underset{\sim}{\hat{\theta}} = (\hat{\theta}_1,\ldots,\hat{\theta}_k)$ obtained from tests conducted at k distinct accelerated stress levels is based on the distribution of $\hat{\theta}_i$, $i=1,2,\ldots,k$, and is given by

$$L(\underset{\sim}{\theta}/\underset{\sim}{\hat{\theta}}) = \left[\prod_{i=1}^{k} \frac{(r_i/\theta_i)^{r_i}(\hat{\theta}_i)^{r_i-1}}{\Gamma(r_i)}\right] \exp\left[-\sum_{i=1}^{k} r_i\hat{\theta}_i/\theta_i\right].$$

A likelihood in terms of C and P may be obtained by substituting $\theta_i = C/V_i^P$, $i=1,\ldots,k$ in the above expression. A more convenient

parameterization is obtained by slightly modifying the power rule to

$$\theta_i = C/(V_i/\overline{V})^P ,$$

where

$$\overline{V} = \prod_{i=1}^{k} V_i^{\,r_i \Big/ \sum_{i=1}^{k} r_i} . \qquad (3.2.1)$$

This modification results in a different value for C (in effect, a change in the proportionality constant), while leaving the value of P unaffected. The likelihood is then

$$L(C,P/\underset{\sim}{\hat{\theta}}) = \left\{ \prod_{i=1}^{k} \frac{\left[\frac{r_i(V_i/\overline{V})^P}{C}\right]^{r_i} (\hat{\theta}_i)^{r_i-1}}{\Gamma(r_i)} \right\} \exp\left[- \sum_{i=1}^{k} \frac{r_i\hat{\theta}_i(V_i/\overline{V})^P}{C} \right] . \qquad (3.2.2)$$

In this case the modified power rule gives the likelihood a more circular shape than would be obtained using $\theta_i = C/V_i^P$ and also has the effect of making the likelihood easier to manage computationally.

The usual Bayesian approach requires that a prior distribution $\Pi(C,P)$ be specified in order to express a state of knowledge regarding C and P before data is obtained. Given the data $\underset{\sim}{\hat{\theta}} = (\hat{\theta}_1,\ldots,\hat{\theta}_k)$ and a prior distribution $\Pi(C,P)$, a posterior distribution for (C,P) may be obtained from the relationship

$$p(C,P/\underset{\sim}{\hat{\theta}}) \propto L(C,P/\underset{\sim}{\hat{\theta}})\, \Pi(C,P) .$$

The viewpoint held by many Bayesians (e.g., Box and Tiao [3], p. 20) maintains that a statistical inference problem is regarded as solved when an appropriate probability statement can be made regarding the state of nature (i.e., the unknown parameters) in question. Thus, the posterior distribution $p(C,P/\underset{\sim}{\hat{\theta}})$ provides the solution to the problem of what may be inferred about the parameters C and P from the data $\underset{\sim}{\hat{\theta}}$ and a relevant state of prior knowledge represented by $\Pi(C,P)$.

It would be reasonable to assign a continuous prior distribution for C and P , since C and P are defined on a continuous parameter space. This presents difficulties in the sense that the integrals that must be evaluated in order to normalize the posterior distribution do not possess convenient closed form solutions. In such cases, it is usual to resort to numerical integration. As an alternative to numerical integration we consider a simpler numerical procedure. This procedure involves a discretization of the prior distribution.

For our problem, this procedure assumes that prior knowledge regarding C and P may be expressed in terms of a discrete bivariate probability mass function. The process of obtaining a discrete prior for (C,P) may be considered consistent with usual Bayesian procedures in the sense that the prior may, in effect, be specified before observing the data. For instance, a continuous distribution could be selected as representing prior knowledge regarding (C,P) and then appropriately discretized. For computational efficiency it is most economical to discretize the prior only over certain regions of the parameter space which are dictated by the available data. That is, after observing the data a region R of the parameter space is identified for which the likelihood is appreciably non-zero and the prior discretized over R . The general form of R is

$$R = \{(C,P): \alpha \le C \le \beta, \quad \gamma \le P \le \delta\} ,$$

where α, β, γ, δ are boundary points such that outside R the likelihood is negligible. The subset of the parameter space outside R is therefore not relevant to the realized likelihood and thus little or nothing is lost by excluding it from consideration.

After R has been identified, a discrete prior may be obtained by dividing R into rectangular subsets and assigning probability mass to each subset. This requires that intervals for C and P , say Δ_1 and Δ_2 , respectively, be chosen so that the subsets defined by

$$R_{mn} = \{(C,P): \alpha+m\Delta_1 \le C \le \alpha+(m+1)\Delta_1, \quad \gamma+n\Delta_2 \le P \le \gamma+(n+1)\Delta_2\} ,$$

$$m=0,1,\ldots,M , \quad n=0,1,\ldots,N ,$$

provide a suitable partition of R. The probability mass assigned to R_{mn}, denoted by W_{mn}, is then associated with the mid-point of R_{mn}, $(C_m = \alpha+(m+\frac{1}{2})\Delta_1,\ P_n = \gamma+(n+\frac{1}{2})\Delta_2)$. The discrete prior distribution is then

$$\Pi(C_m,P_n) = W_{mn}\ , \quad m=0,1,\ldots,M\ , \quad n=0,1,\ldots,N\ ,$$

where $\Pi(C_m,P_n)$ may be interpreted as the approximate prior probability that $C = C_m$ and $P = P_n$. The posterior is then

$$p(C_m,P_n/\underset{\sim}{\hat{\theta}}) = \frac{L(C_m,P_n/\underset{\sim}{\hat{\theta}})\Pi(C_m,P_n)}{F}\ , \quad m=0,1,\ldots,M\ , \quad n=0,1,\ldots,N\ ,$$

where the likelihood $L(C_m,P_n/\underset{\sim}{\hat{\theta}})$ is given by Equation (3.2.2), and

$$F = \sum_{m=0}^{M}\sum_{n=0}^{N} L(C_m,P_n/\underset{\sim}{\hat{\theta}})\ .$$

It is difficult to specify what constitutes a suitable partition of R. Generally, the interval lengths Δ_1 and Δ_2 should be of a size to adequately represent the behavior of the likelihood over R. If the intervals are too large, much of the variation in the posterior will be omitted, and if they are too small, the computations will become unwieldy.

After R and Δ_1,Δ_2 have been determined, obtaining a discrete prior distribution based on a continuous distribution is reasonably straightforward. Suppose $f(x,y)$ is the p.d.f. of an absolutely continuous distribution which is an appropriate representation of prior knowledge regarding (C,P). The probability masses for a discrete prior distribution may then be obtained by evaluating the probabilities

$$\Pr[\alpha+m\Delta_1 \le X \le \alpha+(m+1)\Delta_1,\ \gamma+n\Delta_2 \le Y \le \gamma+(n+1)\Delta_2] =$$

$$\int_{\alpha+m\Delta_1}^{\alpha+(m+1)\Delta_1}\int_{\gamma+n\Delta_2}^{\gamma+(n+1)\Delta_2} f(x,y)\,dy\,dx = W_{mn}\ , \quad m=0,1,\ldots,M\ , \quad n=0,1,\ldots,N\ .$$

An alternative method for constructing a discrete prior for (C,P) is to subjectively assign probability masses to subsets of the parameter space. After the region R has been identified, we may assign a subjectively chosen probability mass W_{mn} to each of the subsets R_{mn}. For example, a discrete locally uniform prior over the region R is obtained by assigning a constant probability to each of the subsets R_{mn}. The procedure is quite general in the sense that any values for the W_{mn} may be assigned.

2.1 Example

The results of a simulated accelerated life test were given in Table 2.1. In this section we use this data to illustrate the Bayesian methods presented in this chapter.

We make the assumption here that prior knowledge regarding (C,P) may be expressed in terms of a joint prior distribution which assigns a positive probability for non-negative values of C and P. A joint distribution which satisfies this requirement is the absolutely continuous bivariate exponential (ACBVE) distribution given by Block and Basu [2]. The ACBVE is defined for $0 < x < \infty$, $0 < y < \infty$. In order to have more flexibility in specifying the prior, especially in connection with the location of the observed likelihood, we introduce location parameters μ_1 and μ_2. The density for the shifted form of the ACBVE is then

$$f(x,y) = \begin{cases} \dfrac{\lambda_1 \lambda(\lambda_2+\lambda_{12}) e^{-\lambda_1(x-\mu_1) - (\lambda_2+\lambda_{12})(y-\mu_2)}}{2\lambda - (\lambda_2+\lambda_{12}) e^{-\lambda_1(\mu_2-\mu_1)} - (\lambda_1+\lambda_{12}) e^{-\lambda_2(\mu_1-\mu_2)}} & \text{for } \mu_1 < x < y \\[2ex] \dfrac{\lambda_2 \lambda(\lambda_1+\lambda_{12}) e^{-(\lambda_1+\lambda_{12})(x-\mu_1) - \lambda_2(y-\mu_2)}}{2\lambda - (\lambda_2+\lambda_{12}) e^{-\lambda_1(\mu_2-\mu_1)} - (\lambda_1+\lambda_{12}) e^{-\lambda_2(\mu_1-\mu_2)}} & \text{for } \mu_2 < y < x \, , \end{cases}$$

where

$$\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$$

$$\lambda_1, \lambda_2, \lambda_{12} > 0 \ , \quad \mu_1, \mu_2 \geq 0 \ .$$

By introducing the location parameters we also avoid assigning high prior probability near the origin.

Three contours of posterior distributions, together with the mode $(\hat{C},\hat{P})$, obtained using different prior distributions are shown in Figure 3.1(i) - (vii). The levels of these contours are defined by the following:

$$\text{A:} \quad p(C,P/\underset{\sim}{\theta}) \geq 0.5\ p(\hat{C},\hat{P}/\underset{\sim}{\hat{\theta}})$$

$$\text{B:} \quad p(C,P/\underset{\sim}{\hat{\theta}}) \geq 0.25 p(\hat{C},\hat{P}/\underset{\sim}{\hat{\theta}})$$

$$\text{C:} \quad p(C,P/\underset{\sim}{\hat{\theta}}) \geq 0.05 p(\hat{C},\hat{P}/\underset{\sim}{\hat{\theta}}) \ .$$

That is, the probability associated with values of (C,P) within the A contour is at least 0.5 as large as the probability associated with the mode; within B, at least 0.75 as large as the mode; and within C, at least 0.95 as large as the mode. Using the bivariate normal approximation, these contours correspond roughly to the boundaries of the 50, 75, and 95% highest posterior density (H.P.D.) regions, respectively (see Box and Tiao [3], pp. 122-125).

The posterior contours for a uniform prior are shown in Figure 3.1(i). The posterior contours shown in Figure 3.1(ii) - (vii) were obtained using ACBVE priors with parameter values given in Table 3.1. In each case, $\alpha = 0.0285$, $\gamma = 2.225$, $\Delta_1 = 0.001$, $\Delta_2 = 0.05$, $M = 29$, and $N = 43$. The plots in Figure 3.1(i) - (vii) are similar so that for the cases considered, the priors have little effect on the posterior contours. Also, for each case, the mode $(\hat{C},\hat{P})$ is $(0.039, 3.05)$. The similarity of these results may be accounted for by the dominance of the realized likelihood in a small region of the parameter space and the lack of variation of the ACBVE priors in this region. We emphasize here that these results apply only for the data analyzed and that another set of data would yield different results.

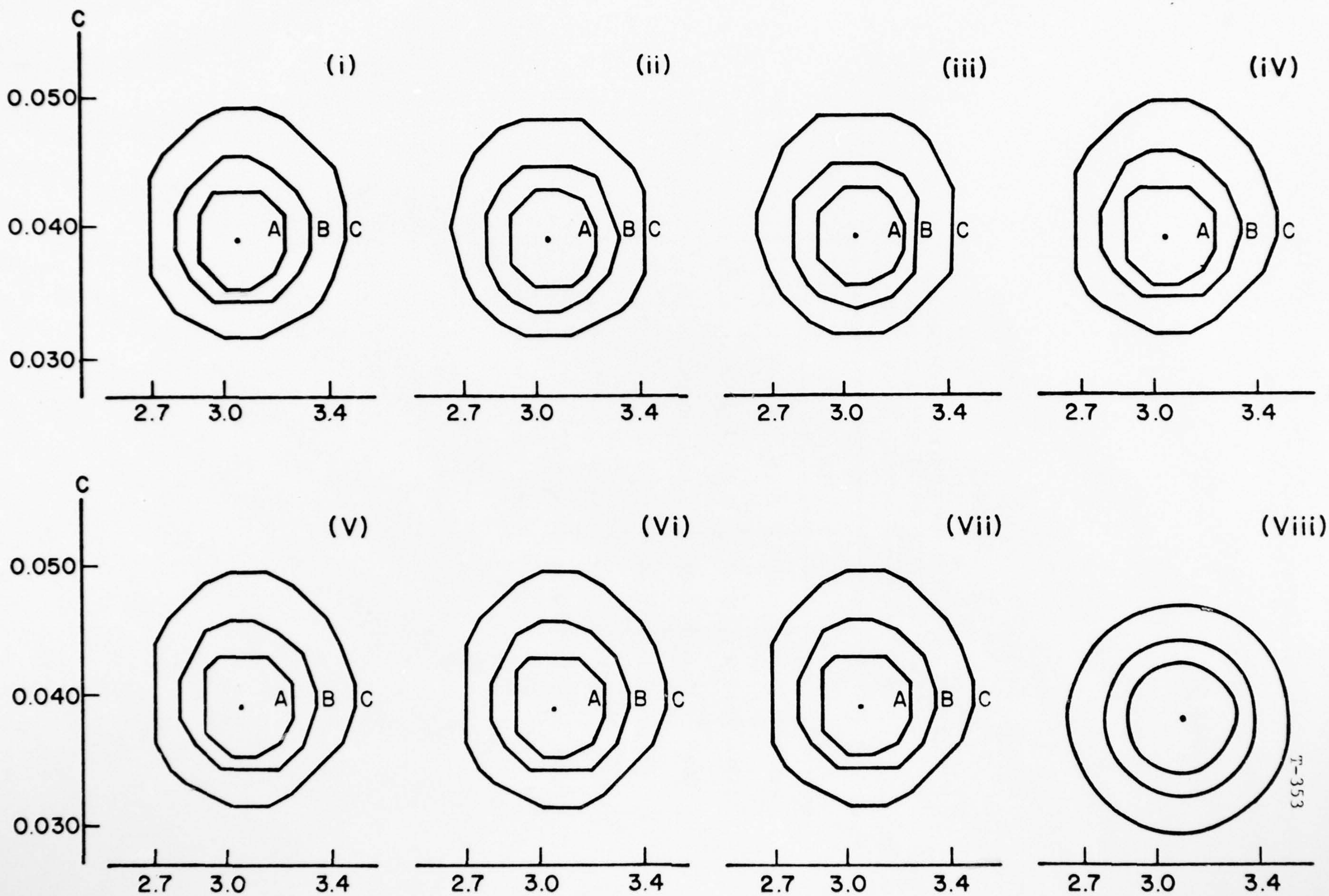

Figure 3.1(i) - (vii)--Contours of the posterior distribution of (C,P) for different prior distributions: Singpurwalla [19] data. (viii)--Bivariate normal contours

TABLE 3.1

PARAMETER VALUES FOR THE BIVARIATE EXPONENTIAL PRIORS USED TO OBTAIN THE POSTERIOR CONTOURS SHOWN IN FIGURE 3.1(i) - (vii)

	λ_1	λ_2	λ_{12}	μ_1	μ_2
(ii)	25.0	0.20	0.18	0.0285	2.225
(iii)	15.0	0.25	0.20	0.0285	2.225
(iv)	0.5	0.08	0.10	0.0285	2.225
(v)	0.5	0.08	0.10	0.0	2.225
(vi)	0.5	0.08	0.10	0.0285	0.0
(vii)	0.5	0.08	0.10	0.0	0.0

For comparison, the exact 50, 75, and 95% contours and the mode of the bivariate normal distribution corresponding to this problem are shown in Figure 3.1(viii). Following a theorem of Lindley ([10], p. 130), posterior distributions are asymptotically approximately multivariate normal with means equal to the maximum likelihood estimates and variance-covariance matrix equal to the information matrix evaluated at the maximum likelihood estimates. Strictly stated, Lindley's theorem does not apply in this case since we have a priori restricted (C,P) to positive values. However, the bivariate normal contours based on the maximum likelihood estimates for (C,P) are such that negative values have negligible probability. The contours of the posterior distributions are roughly similar in shape and location to the bivariate normal contours. The asymptotic normality of the posterior depends on the sample size which in this case corresponds to the number of stress levels and the number of failures per stress level. Thus, as the number of stress levels and the number of failures per stress level increase, we would expect the agreement of the posterior contours with the bivariate normal contours to be enhanced.

2.2 Prediction at Use Stress

Since we assume that the power rule relationship holds at use stress conditions, a particular predicted value for θ_u is obtained as

$$\theta_{u(m,n)} = C_m (V_u/\overline{V})^{-P_n} ,$$

where V_u is use stress and $\overline{V}$ is given by Equation (3.2.1). Given the discrete posterior distribution for (C,P), the probability associated with $\theta_{u(m,n)}$ is

$$p\left[\theta_{u(m,n)} = C_m (V_u/\overline{V})^{-P_n} \Big/ \hat{\underset{\sim}{\theta}}\right] = p(C_m, P_n/\hat{\underset{\sim}{\theta}}) .$$

Thus, a predictive distribution for θ_u may be obtained based on the posterior distribution of (C,P). It also follows that an approximate 95% highest posterior density (H.P.D.) region for θ_u may be determined

based on the 95% H.P.D. region of $p(C_m, P_n / \underset{\sim}{\hat{\theta}})$. That is, the values of $\theta_{u(m,n)}$ for (C_m, P_n) such that $p(C_m, P_n / \underset{\sim}{\hat{\theta}}) \geq 0.05\ p(\hat{C}, \hat{P} / \underset{\sim}{\hat{\theta}})$ constitute the set of θ_u values within the approximate 95% H.P.D. region for θ_u.

By grouping the predicted values for θ_u within the approximate 95% H.P.D. region and summing the probabilities associated with each group, we may obtain a distribution for θ_u. In order to arrange the predicted values for θ_u in groups, we define

$$\theta_{u(j)} = \theta_{u(0)} + j\Delta/2 \ , \quad j=1,2,\ldots,J \ ,$$

where $\theta_{u(0)}$ is the lower limit for values of θ_u within the approximate 95% H.P.D. region, Δ is an arbitrarily selected interval size, and J is the number of intervals. We note that J and Δ are selected so that the largest value predicted for θ_u is between $\theta_{u(J)} + \Delta/2$ and $\theta_{u(J)} - \Delta/2$. The probability associated with $\theta_{u(J)}$ is obtained as

$$p(\theta_{u(j)} / \underset{\sim}{\hat{\theta}}) = \sum_{(m,n)_j} p(\theta_{u(m,n)} / \underset{\sim}{\hat{\theta}}) \ ,$$

where

$$(m,n)_j = \{(m,n) : \theta_{u(m,n)} \leq \theta_{u(j)} + \Delta/2 \ \text{ and } \ \theta_{u(m,n)} \geq \theta_{u(j)} - \Delta/2\} \ ,$$

$$j=1,\ldots,J \ .$$

Some distributions of predicted values for θ_u at $V_u = 7$ are shown in Figure 3.2. These were obtained from the posterior distributions with contours shown in Figure 3.1(i) - (iv). The labels (i) - (iv) in Figure 3.2 correspond to the cases shown in Figure 3.1(i) - (iv). As would be expected from the similarity of the posterior contours for (C,P), the prediction distributions for θ_u are quite similar.

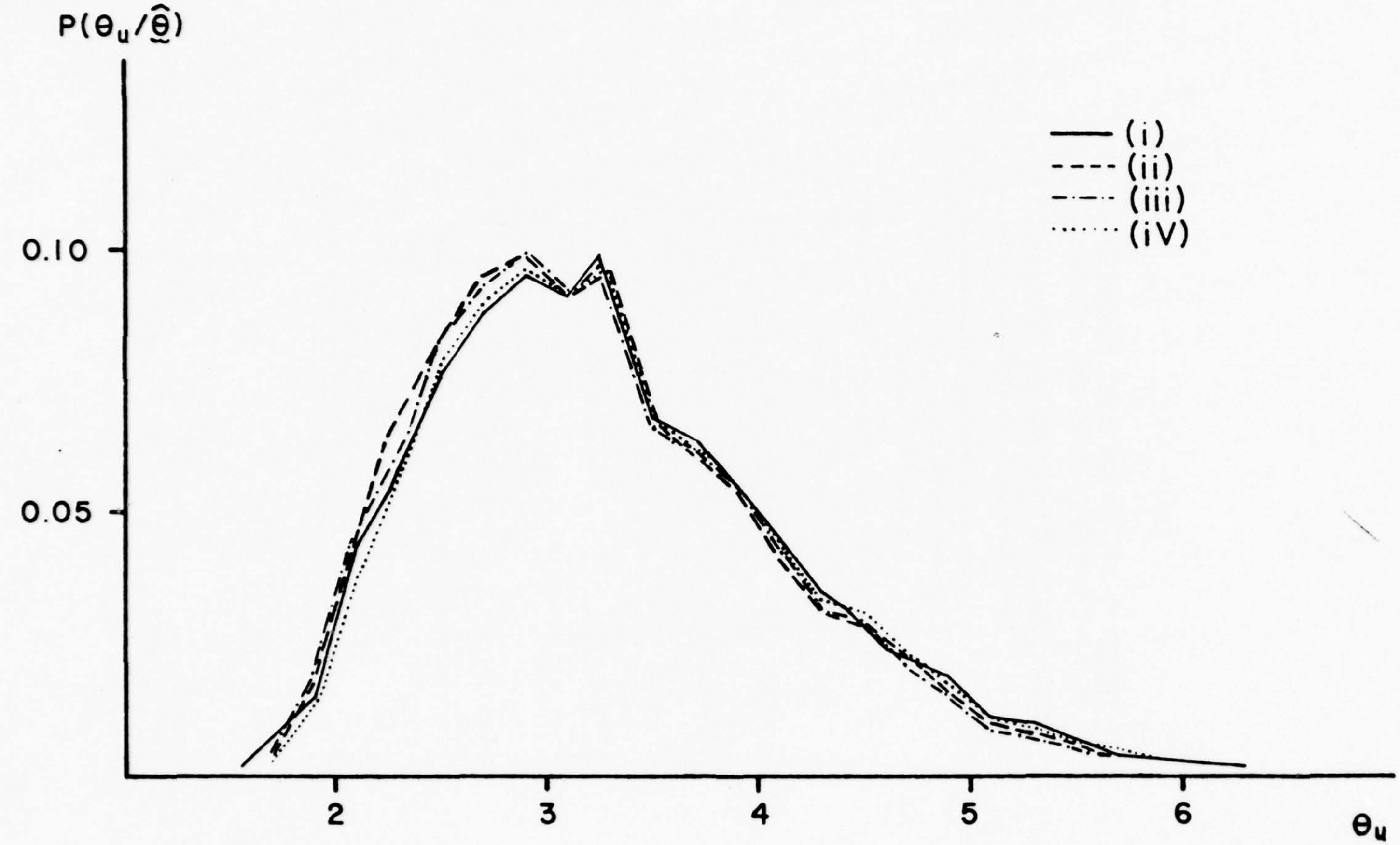

Figure 3.2--Distributions of predicted values for θ_u at $V_u=7$ obtained from the posterior distributions with contours shown in Figure 3.1(i) - (iv)

TABLE 3.2

APPROXIMATE 95% PROBABILITY LIMITS FOR θ_u AT $V_u=7$ OBTAINED FROM THE POSTERIOR DISTRIBUTIONS WITH CONTOURS SHOWN IN FIGURE 3.1(i) - (vii)

	Approximate 95% Probability Limits for θ_u
(i)	1.72 - 6.31
(ii)	1.72 - 6.02
(iii)	1.72 - 6.02
(iv)	1.72 - 6.16
(v)	1.72 - 6.16
(vi)	1.72 - 6.16
(vii)	1.72 - 6.16

The approximate 95% probability limits for θ_u are obtained as the largest and smallest $\theta_{u(m,n)}$ such that $p(\theta_{u(m,n)}/\hat{\underset{\sim}{\theta}}) \geq$ 0.05 $p(\hat{C},\hat{P}/\hat{\underset{\sim}{\theta}})$. The approximate 95% probability limits for θ_u based on the posterior distributions with contours shown in Figure 3.2 are shown in Table 3.2. The values of these probability limits, for the different cases, are the same or very close.

3. The Weibull Case

In this section we consider the use of the power rule with a two-parameter Weibull distribution. The cumulative distribution of the Weibull may be written as

$$F(t;\ \theta_i,b) = 1 - e^{-(t/\theta_i)^b} ,$$

where the scale parameter θ_i is assumed to be a function of the ith stress level. The Weibull is useful as a failure distribution due to its well known flexibility in reflecting the failure characteristics of components and devices.

We consider the power rule as a re-parameterization for the Weibull scale parameter in the following form:

$$\theta_i = 1/CV_i^P ,$$

where V_i describes the stress and C and P are unknown parameters. In this section we use a Bayesian approach for inference concerning the power rule parameters.

When the shape parameter b is known, the Weibull c.d.f. may be written

$$F(x;\ \eta_i) = 1 - e^{-x/\eta_i} ,$$

where $x = t^b$ and $\eta_i = \theta_i^b$. Thus, x has an exponential distribution with scale parameter η_i . An estimate of η_i may be obtained from life

tests conducted as described in Section 2 of this chapter. If the ith test is terminated after r_i $(\le n_i)$ failures occur, a minimum variance, unbiased efficient and sufficient estimator of η_i is (Epstein and Sobel [6])

$$\hat{\eta}_i = \sum_{j=1}^{r_i} \frac{t_{ij}^b + (n_i - r_i) t_{ir_i}^b}{r_i} ,$$

where t_{ij} , $j=1,\ldots,r_i$ are the observed failure times at the ith stress level. The distribution of $\hat{\eta}_i$ is a gamma distribution with scale parameter (r_i/η_i) and shape parameter r_i .

The approach for obtaining posterior distributions for the power rule parameters in this case parallels that of the previous sections for the exponential distribution. In constructing a likelihood in terms of C and P we modify the power rule to

$$\theta_i = 1/C(V_i/\bar{V})^P ,$$

where $\bar{V}$ is given by Equation (3.2.1). In this case, the likelihood is conditional on the data $\hat{\underset{\sim}{\eta}} = (\hat{\eta}_1,\ldots,\hat{\eta}_k)$ and the value of b . That is,

$$L(C,P/\hat{\underset{\sim}{\eta}},b) = \left\{ \prod_{i=1}^{k} \frac{\left[r_i C^b (V_i/\bar{V})^{Pb}\right]^{r_i} (\hat{\eta}_i)^{r_i - 1}}{\Gamma(r_i)} \right\} \bullet \exp\left[- \sum_{i=1}^{k} r_i \hat{\eta}_i C^b (V_i/\bar{V})^{Pb} \right] .$$

The use of the modified power rule in this case results in the likelihood being computationally easier to manage.

A posterior distribution for C and P may be obtained using the relationship $p(C,P/\hat{\underset{\sim}{\eta}},b) \propto L(C,P/\hat{\underset{\sim}{\eta}},b)\Pi(C,P)$, where $\Pi(C,P)$ is an appropriate prior distribution. Using the discrete procedure, the posterior is

$$p(C_m,P_n/\hat{\underset{\sim}{\eta}},b) = \frac{L(C_m,P_n/\hat{\underset{\sim}{\eta}},b)\Pi(C_m,P_n)}{F} , \quad m=0,1,\ldots,M , \quad n=0,1,\ldots,N .$$

where $\Pi(C_m, P_n)$ is an appropriately discretized prior distribution and $F = \sum_{m=0}^{M} \sum_{n=0}^{N} L(C_m, P_n / \hat{\eta}, b) \Pi(C_m, P_n)$. Procedures for obtaining $\Pi(C_m, P_n)$ and its interpretation given in previous sections dealing with the exponential case also apply in the Weibull case under consideration here.

3.1 Example

Some accelerated life test results published by Nelson [15] are shown in Table 3.3. The data consists of breakdown times of an insulating fluid under different voltage stresses. We assume that the failure times have a Weibull distribution with scale parameter θ_i reparameterized in terms of the stress by the power rule, and shape parameter b independent of the stress.

We assume here, as with the exponential case, that prior knowledge regarding (C,P) may be expressed in terms of a joint prior which assigns positive probability for non-negative values of C and P . Thus, the absolutely continuous bivariate exponential (ACBVE) distribution used as a prior in the exponential failure case may also be used here.

Contours of some posterior distributions for (C,P) , together with their modes, are shown in Figure 3.3(i) - (vi). The contours labeled A, B, C are the approximate 50, 75, and 95% H.P.D. regions, respectively, obtained as described in Section 2.1 of this chapter. The posterior contours shown in Figure 3.3(i) - (ii) were obtained using a uniform prior. The posterior contours shown in Figure 3.3(iii) - (vi) were obtained using ACBVE prior distributions with parameters given in Table 3.4. For each case shown in Figure 3.3(i) - (vi) the values $\alpha = 0.031$, $\gamma = 11.375$, $\Delta_1 = 0.002$, $\Delta_2 = 0.25$, $M = 47$, and $N = 49$ were used. The two values of b used to obtain the posterior contours shown in Figure 3.3(i) - (vi) are $b = 1.0$, which is equivalent to assuming that the failure times follow an exponential distribution, and $b = 0.808$

TABLE 3.3

OBSERVED TIMES TO FAILURE OF AN INSULATING FLUID AT DIFFERENT VOLTAGE STRESS LEVELS (FROM NELSON [15])

26 kV	28 kV	30 kV	32 kV	34 kV	36 kV	38 kV
5.79	68.85	7.74	0.27	0.19	0.35	0.09
1579.40	108.29	17.05	0.40	0.78	0.59	0.39
2323.66	110.29	20.46	0.69	0.96	0.96	0.47
	426.07	21.02	0.79	1.31	0.99	0.73
	1067.53	22.66	2.75	2.78	1.69	0.74
		43.40	3.91	3.16	1.97	1.13
		47.30	9.88	4.15	2.07	1.40
		139.06	13.95	4.67	2.58	2.38
		144.11	15.93	4.85	2.71	
		175.88	27.80	6.50	2.90	
		194.88	53.24	7.35	3.67	
			82.85	8.01	3.99	
			89.28	8.27	5.35	
			100.58	12.06	13.77	
			215.10	31.75	25.50	
				32.52		
				33.91		
				36.71		
				72.89		

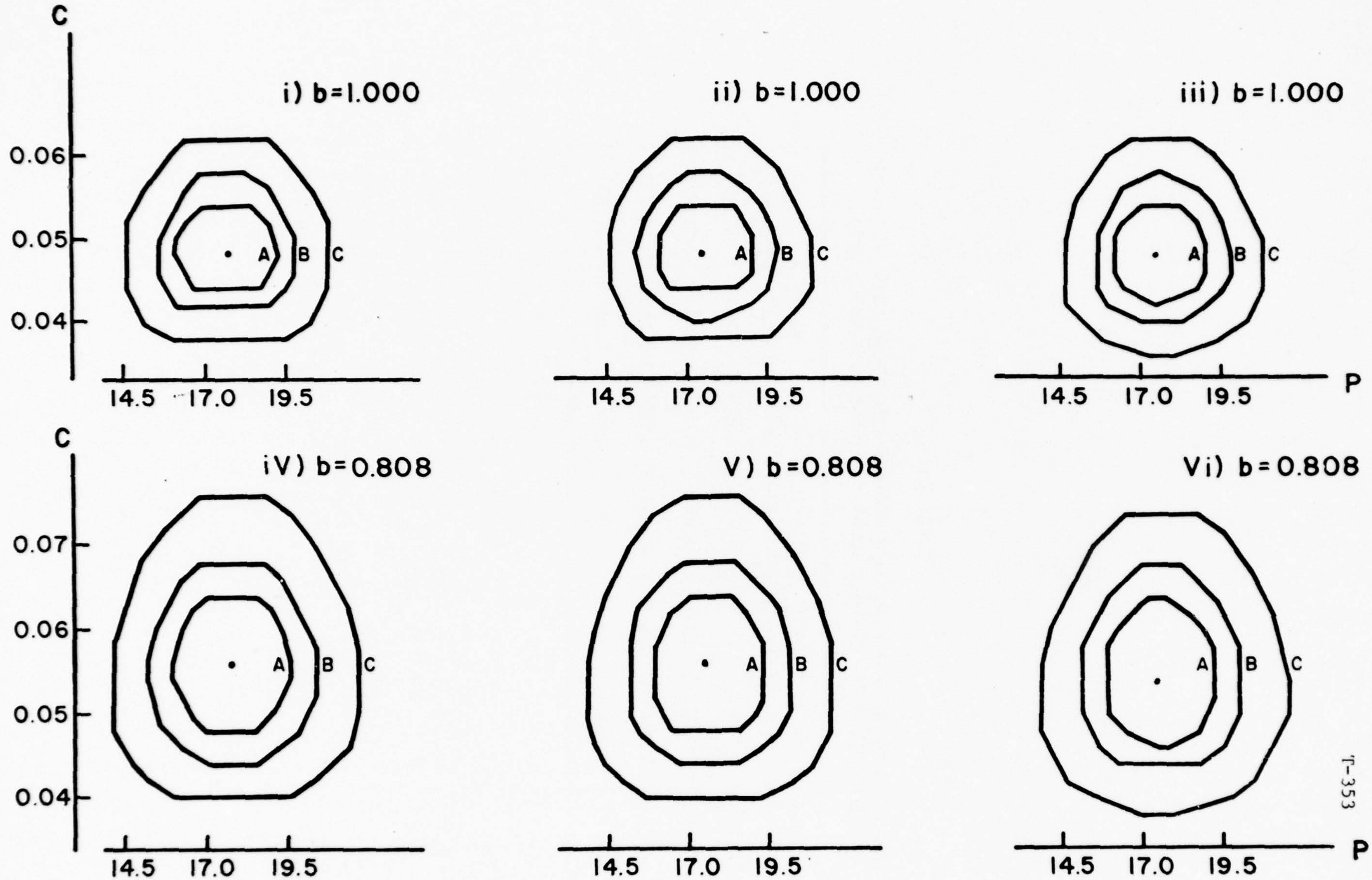

Figure 3.3(i) - (vi)--Contours of the posterior distributions of (C,P) obtained using different priors with the same value of b : Nelson [15] data

TABLE 3.4

PARAMETER VALUES FOR BIVARIATE EXPONENTIAL PRIORS USED TO OBTAIN THE POSTERIOR CONTOURS SHOWN IN FIGURE 3.3(iii) - (vi)

	b	λ_1	λ_2	λ_{12}	μ_1	μ_2
(iii)	1.000	5.0	0.05	0.02	0.031	11.875
(iv)	0.808	5.0	0.05	0.02	0.031	11.875
(v)	1.000	18.0	0.04	0.02	0.031	11.875
(vi)	0.808	18.0	0.04	0.02	0.031	11.875

is the value of the least squares estimate obtained by Nelson [15]. The shape and location of the contours of the posterior distributions shown in Figure 3.3 suggest, for the cases considered, that the choice of b has a greater effect on the posterior than the choice of priors.

The Bayesian framework has the advantage of allowing for the adjustment of the posterior in terms of the shape parameter b. That is, by conditioning $p(C,P/\hat{\underset{\sim}{\eta}},b)$ on different values for b, we may obtain an indication of how sensitive inferences concerning (C,P) are to changes in b. Three contours of posterior distributions of (C,P), together with their modes, for different choices of b, are shown in Figure 3.4. In each case, uniform priors were used with $\alpha = 0.031$, $\gamma = 11.375$, $\Delta_1 = 0.002$, $\Delta_2 = 0.25$, $M = 47$, and $N = 49$. The choice of $b = 1.0$ and $b = 0.808$ was described above, $b = 0.7709$ and $b \approx 0.6962$ were obtained using the maximum likelihood estimation procedures of Singpurwalla and Al-Khayyal [21], $b = 0.925$ and 0.610 are roughly the 95% confidence limits obtained by Nelson [12], and $b = 0.5$ was arbitrarily chosen.

The contours shown in Figure 3.4 give an indication of how inferences about (C,P) are affected by the choice of b. As b changes within a small range, the shape of the posterior contours change significantly. An assumption of exponentiality ($b = 1.0$) produces a rather different result than is obtained using the least squares estimate ($b = 0.808$) or the maximum likelihood estimates ($b = 0.7709$ or 0.6962). In particular, the dispersion of the joint posterior distribution of (C,P) increases markedly as b decreases from 1.0 to 0.5.

Changes in the joint posterior mode for the different values of b are shown in Table 3.5. With the exception of $b = 0.5$, different values for $\hat{C}$ account for changes in the mode of the joint posterior.

The effect of changes in b on the distributions of C and P may be examined by obtaining the marginal posterior distributions conditioned on b :

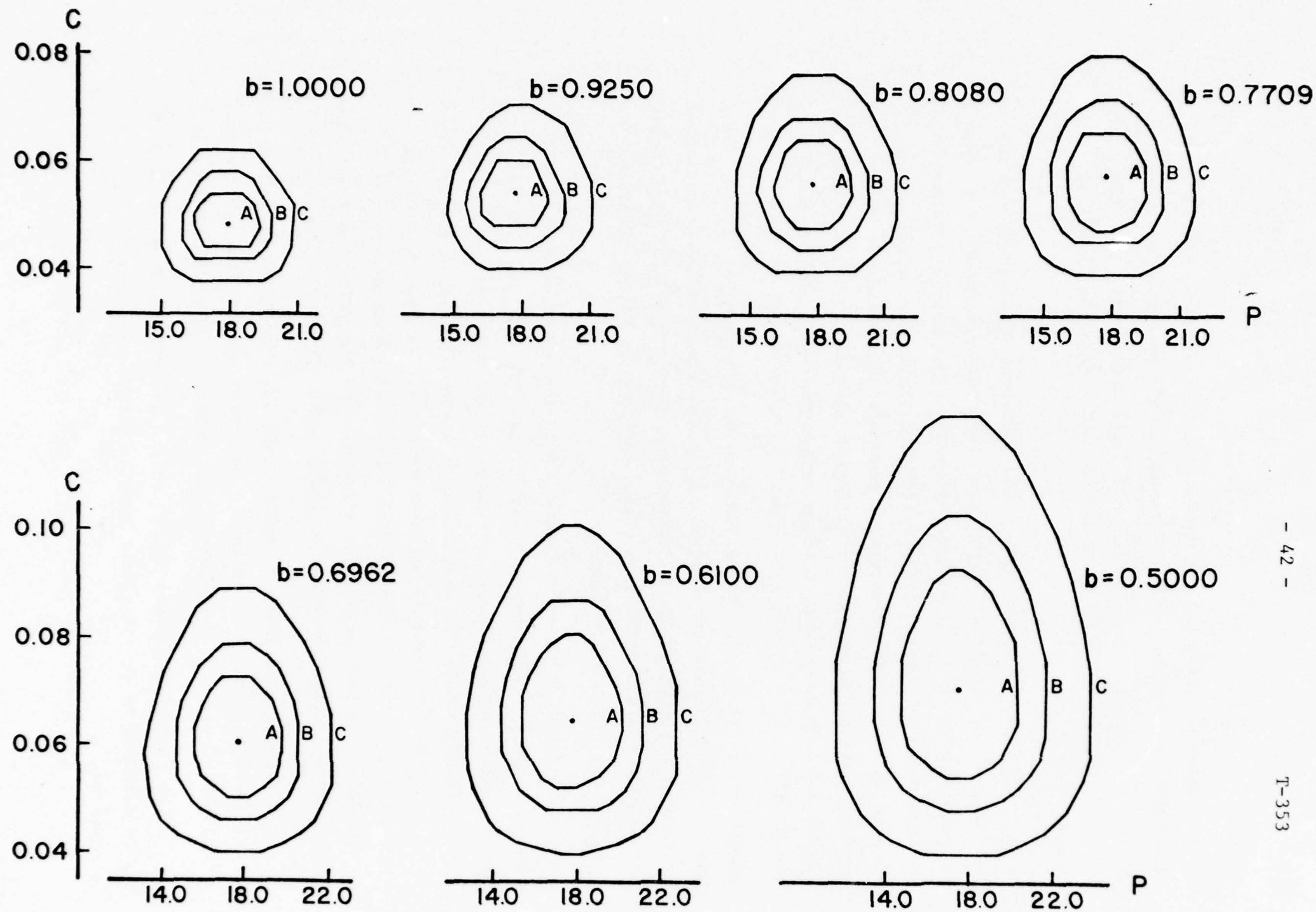

Figure 3.4--Contours of the posterior distribution of (C,P) for various values of b : Nelson [15] data

TABLE 3.5

VALUES OF THE MODE $(\hat{C},\hat{P})$, $\hat{\theta}_u = \left[\hat{C}(V_u/\overline{V})^{\hat{P}}\right]^{-1}$ AND APPROXIMATE 95% PROBABILITY LIMITS FOR θ_u AT $V_u = 20$ AS A FUNCTION OF b, OBTAINED FROM THE POSTERIOR DISTRIBUTIONS WITH CONTOURS SHOWN IN FIGURE 3.4

b	$\hat{C}$	$\hat{P}$	$\hat{\theta}_u$	Approximate 95% Probability Limits for θ_u
1.0000	0.048	17.75	149,442	32,000 - 731,000
0.9250	0.052	17.75	137,946	28,000 - 792,000
0.8080	0.056	17.75	128,093	22,000 - 936,000
0.7709	0.058	17.75	123,676	19,000 - 1,020,000
0.6962	0.060	17.75	119,553	15,000 - 1,156,000
0.6100	0.064	17.75	112,081	10,000 - 1,458,000
0.5000	0.070	17.50	90,429	5,000 - 2,194,000

$$p(C_m/\hat{\eta}, b) = \sum_{n=0}^{N} p(C_m, P_n/\hat{\eta}, b) \ , \qquad m=0,1,\ldots,M$$

and

$$p(P_n/\hat{\eta}, b) = \sum_{m=0}^{M} p(C_m, P_n/\hat{\eta}, b) \ , \qquad n=0,1,\ldots,N \ .$$

Plots of marginal distributions of C and P are shown in Figures 3.5 and 3.6. These were obtained from the joint posterior distributions with the contours shown in Figure 3.4. The marginal distributions also show a marked difference for the different values of b, although the effect on C is more pronounced. The marginal distribution of C changes both dispersion and location as b decreases from 1.0 to 0.5. The dispersion of the marginal distribution of P changes significantly as b decreases from 1.0 to 0.5, but its location remains essentially the same.

We again emphasize in this case that the results obtained here apply only for the particular set of data analyzed. These results indicate that inferences regarding (C,P) are sensitive (i.e., non-robust) to the value of the Weibull shape parameter b. This may be explained by the fact that the estimators $\hat{\eta}_i$ have gamma distributions only when the correct value of b is known. The likelihood used in this case is obtained by taking the joint distribution of $(\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_k)$ conditional on a value for b. If the value selected for b is not close to the correct value, we would reasonably expect inferences based on the gamma assumption to be affected. By conditioning the posterior distributions on different values of b, we obtain an indication of the extent of these effects.

3.2 Prediction at Use Stress

Obtaining a distribution of predicted values of θ_u in the Weibull case follows the same lines as the exponential case discussed in Section 2.2. That is, we assume that a predicted value for θ_u is obtained as

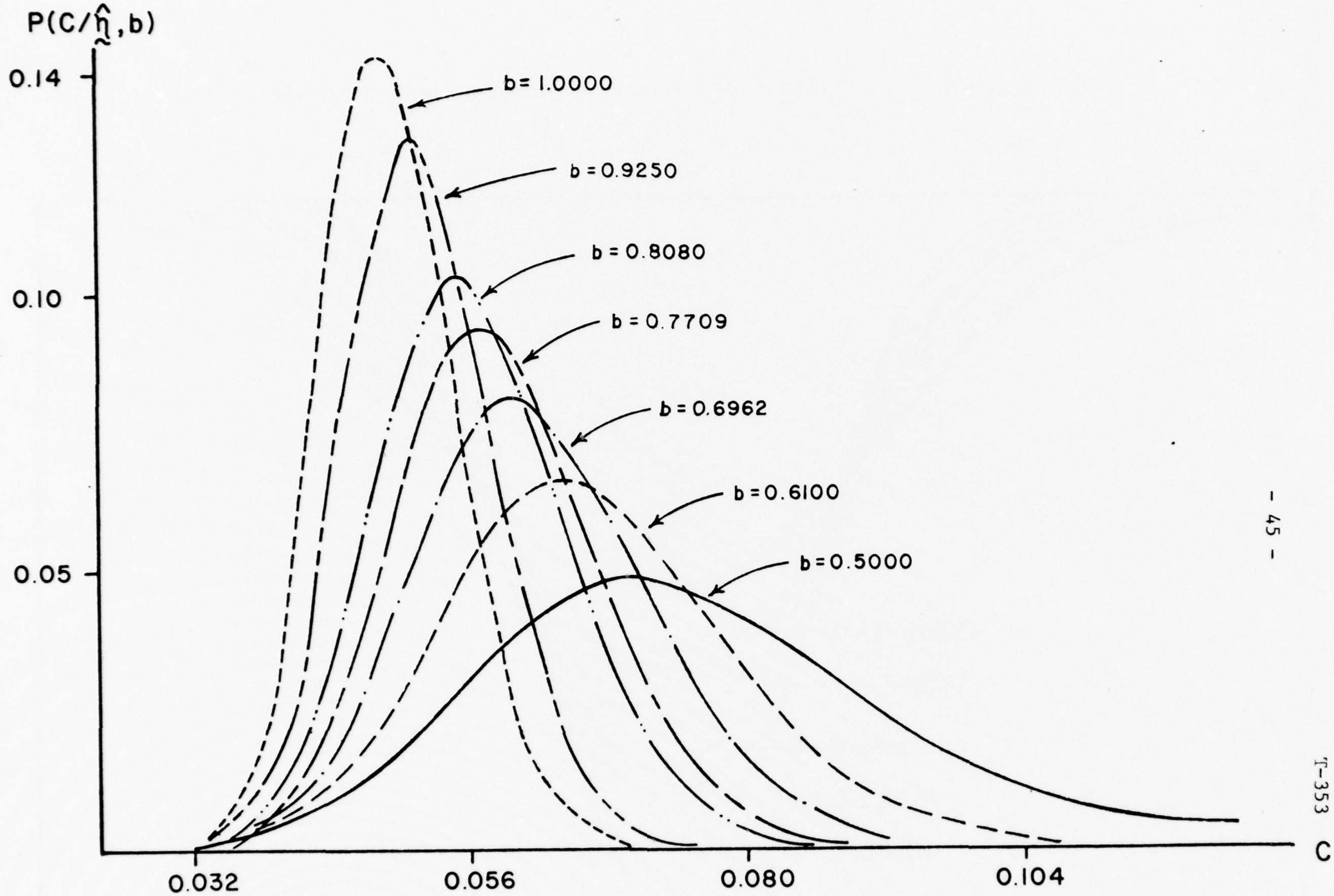

Figure 3.5--Posterior distribution of C for various values of b : Nelson [15] data

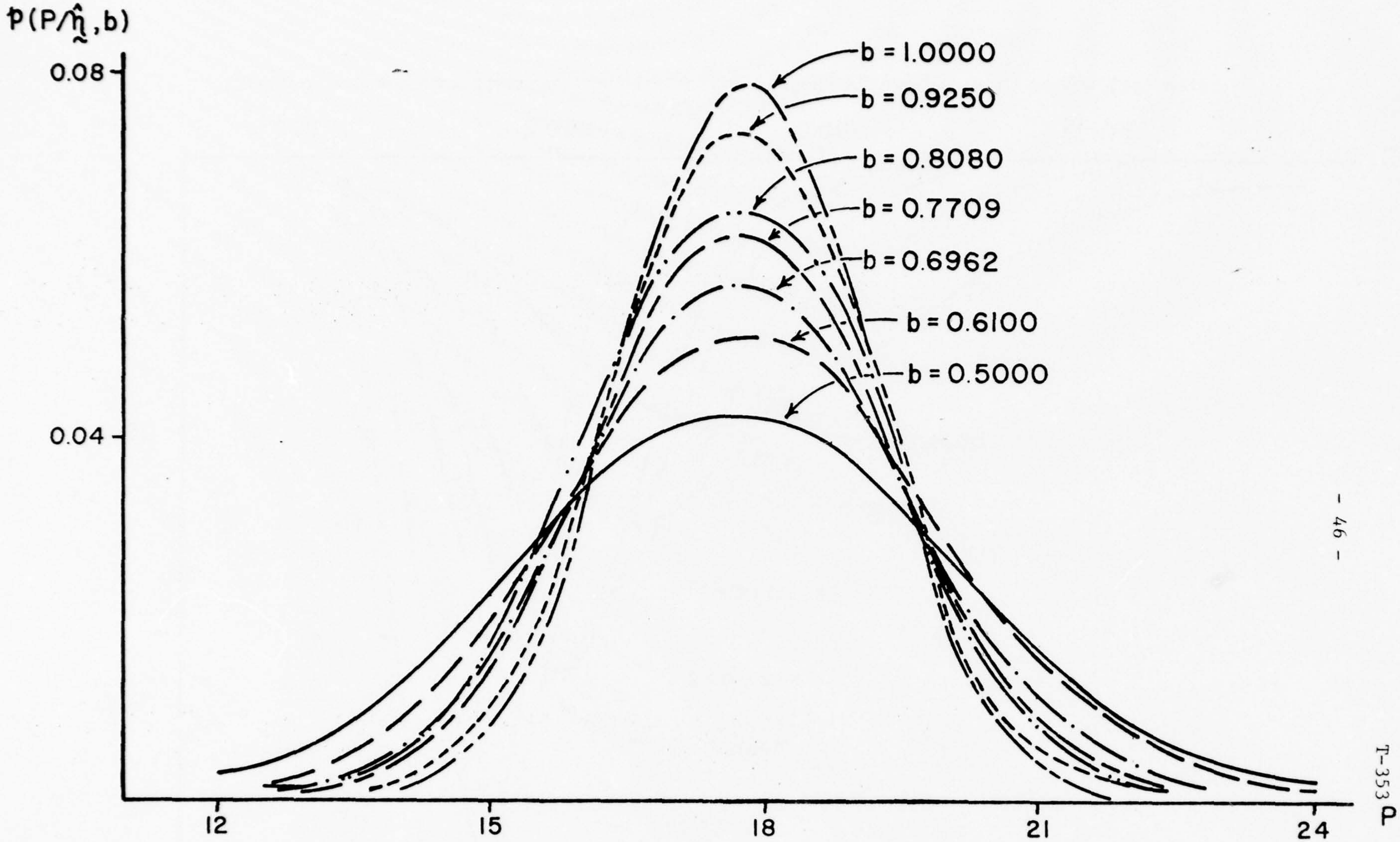

Figure 3.6--Posterior distribution of P for various values of b : Nelson [15] data

$$\theta_{u(m,n)} = \left[C_m (V_u/\bar{V})^{P_n}\right]^{-1} ,$$

where V_u is use condition stress and $\bar{V}$ is given by Equation (3.2.1). The probability associated with $\theta_{u(m,n)}$ is then

$$P(\theta_{u(m,n)}/\hat{\underset{\sim}{\eta}},b) = p(C_m,P_n/\hat{\underset{\sim}{\eta}},b) .$$

The process of obtaining a distribution of predicted values and approximate 95% probability limits for θ_u based on $p(C_m,P_n/\hat{\underset{\sim}{\eta}},b)$ is the same as the exponential case described in Section 2.2.

Some distributions of predicted values for θ_u at $V_u = 20$ are shown in Figure 3.7. These distributions were obtained from the posterior distributions with the contours shown in Figure 3.4. In Table 3.5 are shown the predicted values $\hat{\theta}_u$ obtained using the modal values $(\hat{C},\hat{P})$ and approximate 95% probability limits for θ_u obtained from the posterior distributions shown in Figure 3.4. The dispersions of these prediction distributions, as indicated by the probability limits, are quite sensitive to the choice of b . However, the locations and shapes of the prediction distributions are somewhat similar.

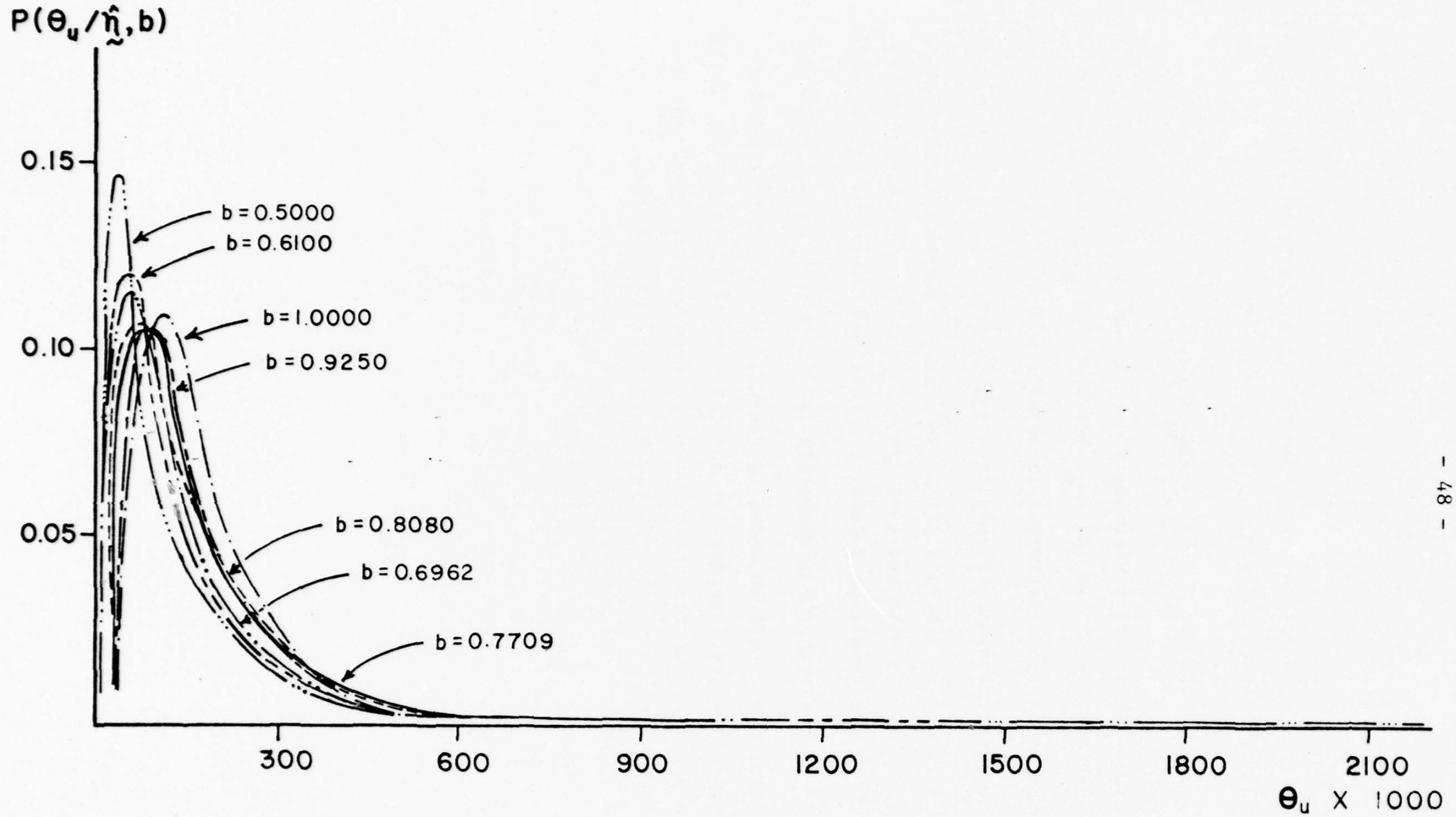

Figure 3.7--Distributions of predicted values for θ_u at V_u=20 for various values of b obtained from the posterior distributions with contours shown in Figure 3.4

CHAPTER IV

SUMMARY AND CONCLUSIONS

In this dissertation we have considered some problems of statistical inference in accelerated life testing. Bayesian and non-Bayesian approaches to these problems were presented. In this chapter the results are summarized and conclusions drawn concerning their applicability.

In the first part of this dissertation we obtained orthogonal least squares estimators for the parameters of three models used to re-parameterize the scale parameter of an exponential distribution: the power rule model, the Arrhenius model, and the Eyring model. We established that these estimators have some convenient and, from the classical or non-Bayesian viewpoint, desirable properties.

(i) They are best linear unbiased and orthogonal for any sample size, i.e., any number of stress levels, greater than the number of parameters being estimated.

(ii) The exact variances and higher moments of the estimators may be readily obtained for any test design.

(iii) The least squares estimators are asymptotically normally distributed. Further, the estimators may be considered approximately normally distributed for tests involving small numbers of stress levels provided a moderate number of failures is observed at each stress.

(iv) The least squares approach avoids the solution of a number of non-linear equations which is required by maximum likelihood estimation. The number of non-linear equations to be solved is equal to the number of parameters being estimated and this could be cumbersome.

Although the least squares estimators are not asymptotically efficient, their variances are exact and only slightly greater than the Cramer-Rao lower bound for any sample size. Also, the least squares estimators have zero covariance for any sample size while the covariance of the maximum likelihood estimates is zero asymptotically.

(v) Best linear unbiased estimators of linear functions of the unknown parameters, such as the log of the exponential scale parameter at use or normal stress, are obtained by substituting the least squares estimators in the function.

In the second part of this dissertation we considered a Bayesian approach for the analysis of the power rule model. Two cases were studied: the power rule used to re-parameterize the scale parameter of an exponential distribution and the scale parameter of a Weibull distribution. In the Bayesian analysis we obtained joint posterior distributions for the two parameters of the power rule. The contours of the posterior distributions shown in this dissertation indicate regions of the parameter space which have high posterior probability of containing the true parameter values. The posterior distributions were obtained using a numerical procedure which involved a discretization of the prior distribution. In both the exponential and Weibull cases, posterior distributions were obtained using uniform and absolutely continuous bivariate exponential (ACBVE) prior distributions. Location parameters were introduced in the ACBVE distribution in order to have more flexibility in specifying the location of the prior distribution with regard to the region of the parameter space for which the likelihood is appreciably non-zero.

For the exponential case, data from simulated accelerated life tests were analyzed with a uniform prior distribution and with ACBVE prior distributions with different parameter values. The contours of the posterior distributions of the power rule parameters were quite similar due to the dominance of the realized likelihood in a relatively small region of the parameter space and the lack of variation of the ACBVE priors in this region. For the Weibull case, data from accelerated life tests of an insulating fluid under voltage stresses were

analyzed. Posterior distributions for the power rule parameters were obtained conditional on different values for the Weibull shape parameter using uniform and different ACBVE prior distributions. The Bayesian approach has the advantage of clearly showing the sensitivity of inference procedures to assumptions about the shape parameter and prior knowledge. Different values of the shape parameter, within a narrow range, were shown to have a marked effect on the posterior distribution of the power rule parameters. That is, using the same prior distribution for the power rule parameters, slight changes in the shape parameter were shown to cause significant changes in the dispersion and location of the posterior distribution. Thus, for the data analyzed, we conclude that inferences about the power rule parameters are not robust with respect to assumptions about the Weibull shape parameter.

The joint posterior distribution of the power rule parameters may be used to obtain inferences about functions of the parameters. The posterior distributions obtained in this dissertation were used to construct distributions of predicted values of the exponential and Weibull scale parameters at use condition or normal stress. In the exponential case the distributions of predicted values for the scale parameter were quite similar since the posterior distributions of the power rule parameters were similar. In the Weibull case the distributions of predicted values for the scale parameter reflected the sensitivity to changes in the Weibull shape parameter of the distribution of the power rule parameters.

The question naturally arises as to which of the approaches presented here, the classical least squares or the Bayesian, should be used in practice. An answer to this question would require that basic philosophical issues regarding the differences between Bayesian and non-Bayesian methods be resolved. In the absence of a generally accepted resolution of these issues the choice of what methods to use depends on the preferences of the analyst. A non-Bayesian in search of a classical solution to the problems considered here might prefer the least squares approach since it results in estimators which have, from the classical viewpoint, desirable properties and presents no computational problems.

On the other hand, a Bayesian would reject the least squares approach, or any other inference procedure that does not result from a likelihood function and a set of prior distributions, as having deficiencies in general cases that may be verified from an objective viewpoint. Indeed, general acceptance of this idea would resolve the question in favor of the Bayesian approach for all inference problems. At the least, the Bayesian approach has some advantages that should be considered:

(i) Most analysts, however neutral philosophically, have some prior knowledge or beliefs about their problems. The Bayesian framework allows for the quantitative expression of this prior knowledge.

(ii) The model or likelihood function through which the data act to modify prior knowledge must be specified explicitly. This requires that careful consideration be given to the process generating the data while also facilitating the exploration of the consequences of alternative assumptions regarding the process.

(iii) In the Bayesian framework inferences are expressed in terms of posterior distributions. This gives a probability interpretation to the inferences that is not possible using the classical confidence intervals.

APPENDIX A

SHAPE FACTORS OF THE LEAST SQUARES ESTIMATES OF THE POWER RULE AND ARRHENIUS MODEL PARAMETERS

In this appendix, we derive expressions for the skewness and kurtosis of the least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ for both the power rule and the Arrhenius model parameters. The skewness and kurtosis, or shape factors, give an indication of the extent to which the distributions of the estimates deviate from normality. Also, since the distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$ approach the normal as k, the number of stress levels, increases, the shape factors give an indication of how rapid the approach is.

The least squares estimates for the power rule and the Arrhenius models, $\hat{\beta}_0$ and $\hat{\beta}_1$, are linear combinations of the observations $\{z_i\}$ which are k independent non-identically distributed log gamma random variables with $Var(z_i) = K_2$, skewness $\gamma_{1,z_i} = K_3/K_2^{3/2}$, and kurtosis $\gamma_{2,z_i} = K_4/K_2^2$, where

$$K_{s+1} = \psi^{(s)}(r_i) , \quad s > 0$$

is the (s+1)st cumulant of a log gamma random variable with shape parameter r_i, and $\psi^{(s)}(\cdot)$ is the sth derivative of the digamma function. Thus, using formulas given by Scheffé ([18], page 332), we may compute the skewness and kurtosis of $\hat{\beta}_0$ and $\hat{\beta}_1$ or any other linear combination of the $\{z_i\}$.

In general, for a linear combination $\xi = \sum_{i=1}^{k} c_i y_i$ of k independent random variables $\{y_i\}$ with variance σ_i^2, skewness $\gamma_{1,i}$ and

kurtosis $\gamma_{2,i}$, the skewness and kurtosis of ξ are

$$\gamma_{1,\xi} = \sum_{i=1}^{k} (\pm)\alpha_i^{3/2}\gamma_{1,i} ,$$

where $(\pm)$ is the sign of c_i , and

$$\gamma_{2,\xi} = \sum_{i=1}^{k} \alpha_i^2\gamma_{2,i} ,$$

where $\alpha_i = c_i^2\sigma_i^2 \Big/ \sum_{i=1}^{k} c_i^2\sigma_i^2$. The α_i are the proportion of the variance of ξ contributed by the ith term, and thus $\sum_{i=1}^{k} \alpha_i = 1$. For $\hat{\beta}_0$, $\alpha_{i,\hat{\beta}_0} = w_i^{-1} \Big/ \sum_{i=1}^{k} w_i^{-1}$ and for $\hat{\beta}_1$, $\alpha_{i,\hat{\beta}_1} = w_i^{-1}x_i^2 \Big/ \sum_{i=1}^{k} w_i^{-1}x_i^2$, where the w_i are given by Equation (2.2.4) and x_i is given by Equation (2.2.7) for the power rule and Equation (2.3.3) for the Arrhenius model.

We may note that the effects of observing a small number of failures at one or more stress levels in an accelerated test may be offset by observing a larger number of failures at other stress levels. That is, since the shape factors of the estimators are weighted averages of the shape factors of the individual observations, the effect of obtaining a few failures at one or more stress levels may be balanced by observing larger numbers of failures at other stress levels.

The following results may be shown to hold if the r_i , the number of failures observed at the ith stress level, are unequal. We will assume, however, that $r_i = r$ for all i since this results in less complicated expressions.

When $r_i = r$, $i=1,\ldots,k$, the variance, skewness and kurtosis of the z_i's are equal and, using the notation $w_i = w$, $\gamma_{i,z_i} = \gamma_1$ and $\gamma_{2,z_i} = \gamma_2$, $i=1,\ldots,k$, it follows that $\alpha_{i,\hat{\beta}_0} = 1/k$, $i=1,\ldots,k$,

and $\gamma_{1,\hat{\beta}_0} = k^{-\frac{1}{2}} \gamma_1$, and $\gamma_{2,\hat{\beta}_0} = k^{-1}\gamma_2$. Clearly, as k is increased with r fixed, the shape factors for $\hat{\beta}_0$ approach the normal distribution value of zero. Using the approximations given by Johnson and Kotz ([8], page 196) for the shape factors of a log gamma random variable with shape parameter r , $\gamma_1 \approx -\left(r - \frac{1}{2}\right)^{-\frac{1}{2}}$ and $\gamma_2 \approx 2\left(r - \frac{1}{2}\right)^{-1}$, it is clear that if k is held fixed, increasing r also causes the shape factors to approach zero.

For $\hat{\beta}_1$, if $r_i = r$, $i=1,\ldots,k$, $\alpha_{i,\hat{\beta}_1} = x_i^2 \Big/ \sum_{j=1}^{k} x_j^2$, $i=1,\ldots,k$, so that $\gamma_{1,\hat{\beta}_1} = \sum_{i=1}^{k} \left(x_i^2 \Big/ \sum_{j=1}^{k} x_j^2\right)^{3/2} \gamma_1$ and $\gamma_{2,\hat{\beta}_1} = \sum_{i=1}^{k} \left(x_i^2 \Big/ \sum_{j=1}^{k} x_j^2\right)^{2} \gamma_2$. Again, using the approximations for γ_1 and γ_2 , it is clear that holding k fixed, increasing r causes the shape factors for $\hat{\beta}_1$ to approach zero. Holding r fixed as $k \to \infty$, the rate of approach to zero of the shape factors for $\hat{\beta}_1$ will be similar to that for $\hat{\beta}_0$ if the $\alpha_{i,\hat{\beta}_1}$ are approximately equal for all i . In many cases of the power rule and the Arrhenius models, however, one or more of the x_i^2 has a limiting value which is disproportionately larger than the others and thus k must be quite large before the corresponding $\alpha_{i,\hat{\beta}_1}$ becomes small, which in turn allows $\sum_{i=1}^{k} \alpha_{i,\hat{\beta}_1}^{3/2}$ and $\sum_{i=1}^{k} \alpha_{i,\hat{\beta}_1}^{2}$ to approach zero. Although in practice k is usually rather small, the shape factors for both $\hat{\beta}_0$ and $\hat{\beta}_1$ can be made reasonably close to the normal distribution values for moderate size r , as shown in Table A.1.

TABLE A.1

SHAPE FACTORS FOR THE ORTHOGONAL LEAST SQUARES ESTIMATORS OF THE POWER RULE AND ARRHENIUS PARAMETERS

	k = 5		k = 10	
r	$\gamma_{1,\hat{\beta}_0}$	$\gamma_{2,\hat{\beta}_0}$	$\gamma_{1,\hat{\beta}_0}$	$\gamma_{2,\hat{\beta}_0}$
10	-.14510	.04211	-.10260	.02105
15	-.11744	.02759	-.08305	.01379
20	-.10127	.02051	-.07161	.01026

r	$\gamma_{1,\hat{\beta}_1}$	$\gamma_{2,\hat{\beta}_1}$
10	-.32444S	.21053K
15	-.26261S	.13793K
20	-.22646S	.10256K

where

r = number of failures observed at each stress level

k = number of stress levels

$$S = \sum_{i=1}^{k} \alpha_{i,\hat{\beta}_1}^{3/2}$$

$$K = \sum_{i=1}^{k} \alpha_{i,\hat{\beta}_1}^{2}$$

$$\alpha_{i,\hat{\beta}_1} = x_i^2 \Big/ \sum_{j=1}^{k} x_j^2 \ , \quad i=1,\ldots,k \ ,$$

where the x_i are defined by Equation (2.2.7) for the power rule and Equation (2.3.3) for the Arrhenius model.

Note: Since $\sum_{i=1}^{k} \alpha_{i,\hat{\beta}_1} = 1$, $S,K < 1$.

APPENDIX B

SHAPE FACTORS OF THE LEAST SQUARES ESTIMATES OF THE EYRING MODEL PARAMETERS

In this appendix, we derive expressions for the skewness and kurtosis of the orthogonal least squares estimates of the Eyring parameters, $\hat{\beta}' = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4)$ given by Equation (2.6.8). The skewness and kurtosis, or shape factors, give an indication of the extent to which the distributions of the estimates deviate from normality. Also, since the distributions of the estimates approach the normal as k, the number of stress levels increases, the shape factors give an indication of how rapid the approach is.

The orthogonal least squares estimates of the Eyring model parameters are linear combinations of the observations $\{z_i\}$ which are k independent non-identically distributed log gamma random variables with $Var(z_i) = K_2$, skewness $\gamma_{1,z_i} = K_3/K_2^{3/2}$ and kurtosis $\gamma_{2,z_i} = K_4/K_2^2$, where

$$K_{s+1} = \;^{(s)}(r_i)\,, \qquad s \quad 0$$

is the $(s+1)^{st}$ cumulant of a log gamma random variable with shape parameter r_i and $\psi^{(s)}(\cdot)$ is the s^{th} derivative of the digamma function. Thus, using formulas given by Sheffé ([18], p. 332), we may compute the skewness and kurtosis of the least squares estimates or any other linear combination of the $\{z_i\}$.

The general formulas for the shape factors of a linear combination of independent random variables are given in Appendix A. For the orthogonal least squares estimates of the Eyring model parameters

$$\gamma_{1,\hat{\beta}_j} = \sum_{i=1}^{k} \alpha_{i,\hat{\beta}_j}^{3/2} \gamma_{1,z_i} \quad , \quad j=1,\ldots,4$$

and

$$\gamma_{2,\hat{\beta}_j} = \sum_{i=1}^{k} \alpha_{i,\hat{\beta}_j}^{2} \gamma_{2,z_i} \quad , \quad j=1,2,3,4$$

where

$$\alpha_{i,\hat{\beta}_1} = w_i^{-1} \Big/ \sum_{j=1}^{k} w_j^{-1} \quad , \quad i=1,\ldots,k$$

$$\alpha_{i,\hat{\beta}_2} = x_{i2}^2 w_i^{-1} \Big/ \sum_{j=1}^{k} x_{j2}^2 w_j^{-1} \quad , \quad i=1,\ldots,k$$

$$\alpha_{i,\hat{\beta}_3} = x_{i3}^2 w_i^{-1} \Big/ \sum_{j=1}^{k} x_{j2}^2 w_j^{-1} \quad , \quad i=1,\ldots,k$$

$$\alpha_{i,\hat{\beta}_4} = x_{i4}^2 w_i^{-1} \Big/ \sum_{j=1}^{k} x_{j4}^2 w_j^{-1} \quad , \quad i=1,\ldots,k$$

$$w_i = \psi^{(1)}(r_i) \quad , \quad i=1,\ldots,k$$

and x_{i2} , x_{i3} , x_{i4} for $i=1,\ldots,k$ are given by Equations (2.6.9), (2.6.10), and (2.6.11), respectively.

The following results, which relate to the behavior of the shape factors of the estimates as the number of stress levels and the number of failures observed at each stress increase, may be shown to hold if the r_i , the number of failures observed at the i^{th} stress level, are unequal. However, assuming $r_i = r$ for all i results in less complicated expressions and causes no loss of generality.

When $r_i = r$, $i=1,\ldots,k$, the variance, skewness and kurtosis of the z_i's are equal. Using the notation $w_i = w$, $\gamma_{1,z_i} = \gamma_1$ and $\gamma_{2,z_i} = \gamma_2$ expressions for the skewness and kurtosis of the estimates are obtained as $\gamma_{1,\hat{\beta}_1} = k^{-\frac{1}{2}}\gamma_1$, $\gamma_{2,\hat{\beta}_1} = k^{-1}\gamma_2$ and

$$\gamma_{1,\hat{\beta}_m} = \gamma_1 \sum_{j=1}^{k} \left(x_{jm}^2 \Big/ \sum_{i=1}^{k} x_{im}^2 \right)^{3/2} ,$$

$$\gamma_{2,\hat{\beta}_m} = \gamma_2 \sum_{j=1}^{k} \left(x_{jm}^2 \Big/ \sum_{i=1}^{k} x_{im}^2 \right)^{2} \quad \text{for} \quad m = 2,3,4 .$$

Clearly, as k is increased with r held fixed, the shape factors for $\hat{\beta}_1$ approach the normal distribution values of zero. Using the approximations given by Johnson and Kotz ([8], p. 196) for the shape factors of a log gamma random variable with shape parameter r, $\gamma_1 \approx -(r - \frac{1}{2})^{-\frac{1}{2}}$ and $\gamma_2 \approx 2(r - \frac{1}{2})^{-1}$, it is clear that if k is held fixed, increasing r also causes the shape factors of $\hat{\beta}_1$ to approach zero. For $\hat{\beta}_m$, $m = 2,3,4$, again using the approximations for γ_1 and γ_2, it is clear that increasing r while holding k fixed causes $\gamma_{1,\hat{\beta}_m}$ and $\gamma_{2,\hat{\beta}_m}$ to approach zero. Holding r fixed as $k \to \infty$, the rate of approach to zero of the shape factors for $\hat{\beta}_m$, $m = 2,3,4$ will be similar to that for the shape factors of $\hat{\beta}_1$ if the $\gamma_{j,\hat{\beta}_m} = x_{jm}^2 \Big/ \sum_{i=1}^{k} x_{im}^2$, $j=1,\ldots,k$, are approximately equal for all j. In many cases of the Eyring model, however, one or more of the x_{jm}^2 has a limiting value which is disproportionately larger than the others and thus k must be quite large before the corresponding $\alpha_{j,\hat{\beta}_m}$ becomes small, which in turn allows $\sum_{j=1}^{k} \alpha_{j,\hat{\beta}_m}^{3/2}$ and $\sum_{j=1}^{k} \alpha_{j,\hat{\beta}_m}^{2}$ to approach zero. Although in practice k is usually rather small, the shape factors for $\hat{\beta}_m$, $m = 1,2,3,4$, can be made reasonably close to the normal distribution values for moderate size r, as shown in Table B.1.

TABLE B.1

SHAPE FACTORS FOR THE ORTHOGONAL LEAST SQUARES ESTIMATORS OF THE EYRING PARAMETERS

r	k = 5: $\gamma_{1,\hat{\beta}_1}$	k = 5: $\gamma_{2,\hat{\beta}_1}$	k = 10: $\gamma_{1,\hat{\beta}_1}$	k = 10: $\gamma_{2,\hat{\beta}_1}$
10	-.1451	.0421	-.1026	.0211
15	-.1174	.0276	-.0831	.0138
20	-.1013	.0205	-.0716	.0103

r	$\gamma_{1,\hat{\beta}_m}$	$\gamma_{2,\hat{\beta}_m}$
10	$-.2244\ S_m$	$.2105\ U_m$
15	$-.2626\ S_m$	$.1379\ U_m$
20	$-.2265\ S_m$	$.1026\ U_m$

r = number of failures observed at each stress level

k = number of stress levels

$$S_m = \sum_{j=1}^{k} \alpha_{j,\hat{\beta}_m}^{3/2} \quad , \quad m = 2,3,4$$

$$U_m = \sum_{j=1}^{k} \alpha_{j,\hat{\beta}_m}^{2} \quad , \quad m = 2,3,4$$

$$\alpha_{j,\hat{\beta}_m} = x_{j,m}^2 \Big/ \sum_{i=1}^{k} x_{im}^2 \quad , \quad j = 1,\ldots,k \quad m = 2,3,4$$

with $x_{j,2}$, $x_{j,3}$, $x_{j,4}$, $j = 1,\ldots,k$ given by Equations (2.6.9), (2.6.10) and (2.6.11), respectively.

Note: $\sum_{j=1}^{k} \alpha_{j,\hat{\beta}_m} = 1$, S_m, $U_m < 1$ for $m = 2,3,4$

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